Filtering Closed Curves

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Abstract—A closed curve in the plane can be described in several ways. We show that a simple representation in terms of radius of curvature versus normal direction has certain advantages. In particular, convolutional filtering of the extended circular image leads to a closed curve. Similar filtering operations applied to some other representations of the curve do not guarantee that the result corresponds to a closed curve. In one case, where a closed curve is produced, it is smaller than the original. A description of a curve can be based on a sequence of smoothed versions of the curve. This is one reason why smoothing of closed curves is of interest.

Index Terms—Curvature; edges, lines, and contours; filtering; image representation; matching; 2-D shape description.

I. INTRODUCTION

Given a simple closed rectifiable curve in the plane, how can it be smoothed to produce a similar closed curve? One can try convolution-like filtering operations on various representations of the curve. In most cases the result is not a closed curve. In the one case that a closed curve is obtained, it is smaller than the original curve, as we shall see. We here propose a new representation that guarantees that convolutional filtering will produce only closed curves.

In machine vision one needs descriptions of the shape of closed contours corresponding to object features. Such descriptions can be built up using a multiscale sequence of smoothed curves. It is useful then to have a way of smoothing curves which preserves closure.

II. REPRESENTATIONS OF CLOSED CURVES

One of the ways to represent a curve in the plane is to give x and y as a function of some parameter that varies monotonically along the curve. In the simplest case this would be the arc-length s, measured along the curve from some arbitrary starting point:

$$x = x(s) \quad \text{and} \quad y = y(s).$$

Now $x(s)$ and $y(s)$ are periodic, with period equal to the perimeter P of the curve. Conversely, any pair of periodic functions that have the same period define a closed curve in the plane (although it may not be a simple curve). For example, a circle of radius R, with center at the origin can be represented by the equations:

$$x(s) = R\cos\left(\frac{s}{2\pi R}\right) \quad \text{and} \quad y(s) = R\sin\left(\frac{s}{2\pi R}\right).$$

Filtering $x(s)$ and $y(s)$ by means of convolution results in closed curves, beause the result is still periodic [1]. Filtering can only attenuate or amplify the frequency components already present, it cannot produce new frequencies.

Manuscript received February 19, 1985; revised November 10, 1985. Recommended for acceptance by S. W. Zucker.

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IEEE Log Number 8609775.

We define a *smoothing filter* as one whose impulse response is nonnegative. It is easy to see that the magnitude of the frequency response of such a filter at any frequency has to be less than or equal to its response at zero frequency. Such a filter can also be normalized so that the integral of its impulse response equals one. In this case the magnitude of the response at any frequency other than zero will be less than one. This means that the amplitude of the x and y components of the curve are attenuated, even at the fundamental frequency $1/P$. Thus the size of the curve is reduced. This may be undesirable if a description of the curve is to be based on a multiscale sequence of smoothed curves.

III. WHEWELL AND CESÁRO FORMS

Among intrinsic equations for a curve in the plane is Whewell's form, which gives a (possibly implicit) relationship between the direction of the normal ψ and the arc-length s along the curve [2]. The direction is measured with respect to some arbitrary reference direction. For the circle given above we have:

$$\psi(s) = s/R.$$

This form has been used in machine vision in shape description [3], as well as in classification [4].

Another related intrinsic equation for a curve in the plane is Cesáro's form, in which curvature κ is related to arc-length s along the curve [2]. Curvature can be defined as the rate of change of the direction of the normal as a function of arc length. That is,

$$\kappa(s) = \frac{d\psi}{ds}.$$

For the circle we find

$$\kappa(s) = 1/R.$$

Note that Whewell's and Cesáro's forms are insensitive to the position of the curve in the plane, unlike the first representation we introduced above.

It may be tempting to apply a convolution filter to $\kappa(s)$ [5]. Unfortunately there is no guarantee that the result will correspond to a closed curve. In general it will not.

IV. EXTENDED CIRCULAR IMAGES

By analogy with the extended Gaussian image, which is a way of representing the shape of a convex three-dimensional object [6], we may define an *extended circular image*. In the extended circular image, one is given the radius of curvature R as a function of normal direction ψ. The integral of the extended circular image over some angular interval is equal to the length of the curve which has normal direction falling in that interval. This can also be expressed as follows:

$$R(\psi) = \lim_{\epsilon\to 0} \frac{s(\psi+\epsilon) - s(\psi-\epsilon)}{2\epsilon} = \frac{ds}{d\psi}.$$

That is, the extended circular image is the derivative of the inverse of the Gauss map.

In the case of the circle we simply have:

$$R(\psi) = R.$$

For a convex curve, the relationship between this new representation and Cesáro's form is

$$R(\psi) = 1/\kappa(s),$$

where ψ in $R(\psi)$ identifies the same point on the curve as does s in $\kappa(s)$.

Polygons can be treated in this fashion. Each side of the polygon is mapped into an impulse of area equal to the length of the side. The angle where this impulse appears is just the normal direction of the corresponding side,

$$R(\psi) = \sum_{i=1}^{n} l_i \delta(\psi - \psi_i),$$

where n is the number of sides, while l_i is the length, and ψ_i the normal direction of the ith side. This result can be obtained by considering each side as the limit of a sector of a circle as the radius of the circle tends to infinity. It is sometimes useful to think of the extended circular image as a mass density distribution along the circumference of a unit circle. In the case of a polygon, this distribution is made up of a number of point masses.

A set of scaled vectors normal to the sides of a closed polygon add up to zero. This gives us the equality:

$$\sum_{i=1}^{n} l_i(\cos \psi_i, \sin \psi_i) = (0, 0).$$

This, in turn, is equivalent to the statement that the center of mass of the point masses on the unit circle must be at the center of the circle.

(One could, by the way, use the tangent direction instead of the normal direction in the definition of the extended circular image. This merely amounts to a rotation of 90 degrees.)

V. The Extended Circular Image of an Ellipse

As an example, consider an ellipse with semimajor axis a and semiminor axis b. If we align the major axis with the x-axis, we can write an implicit equation for this curve:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

It is easier to use a parametric form instead:

$$x = a \cos t \quad \text{and} \quad y = b \sin t.$$

We then have

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = a^2 \sin^2 t + b^2 \cos^2 t,$$

while

$$\sin \psi = \frac{a \sin t}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}$$

and

$$\cos \psi = \frac{b \cos t}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}},$$

so that

$$\frac{d\psi}{dt} = \frac{ab}{a^2 \sin^2 t + b^2 \cos^2 t},$$

and since

$$R(\psi) = \frac{ds}{d\psi} = \frac{ds/dt}{d\psi/dt},$$

we finally have the extended circular image,

$$R(\psi) = \frac{a^2 b^2}{[a^2 \cos^2 \psi + b^2 \sin^2 \psi]^{3/2}}.$$

Using methods developed in the next section, one can now easily recover the Cartesian coordinates,

$$x = \frac{a^2 \cos \psi}{\sqrt{a^2 \cos^2 \psi + b^2 \sin^2 \psi}}$$

and

$$y = \frac{b^2 \sin \psi}{\sqrt{a^2 \cos^2 \psi + b^2 \sin^2 \psi}}.$$

The equation for s is an elliptic integral (of the third kind),

$$s = \int_0^{\psi} \frac{a^2 b^2}{[a^2 \cos^2 \eta + b^2 \sin^2 \eta]^{3/2}}\, d\eta,$$

so it is more complicated to express x and y as functions of s.

As we see next, the extended circular image of a closed curve has no fundamental frequency component. Since filtering cannot introduce new frequency components, this means that the filtered extended circular image of a closed curve corresponds to another closed curve.

VI. Fundamental Theorem

We first show how the equation for a curve, in both Whewell and Cartesian forms, can be recovered from an extended circular image.

Since $R(\psi) = (ds/d\psi)$, the equation of the curve in Whewell form is just

$$s(\psi) = \int_0^{\psi} R(\eta)\, dn.$$

Cartesian coordinates x and y can be recovered with equal ease from $R(\psi)$, by noting that

$$\frac{dx}{ds} = -\sin \psi \quad \text{and} \quad \frac{dy}{ds} = \cos \psi,$$

and so

$$x(s) = x(0) - \int_0^{s} \sin \psi(t)\, dt$$

and

$$y(s) = y(0) + \int_0^{s} \cos \psi(t)\, dt,$$

where $(x(0), y(0))$ is the arbitrary starting point where $s = 0$. For a closed curve of perimeter P, $x(P) = x(0)$ and $y(P) = y(0)$, thus

$$\int_0^{P} \cos \psi(s)\, ds = 0, \quad \text{and} \quad \int_0^{P} \sin \psi(sY)\, ds = 0.$$

If the curve is convex, there will be a monotonic relationship between ψ and s, and so we can change variables and obtain:

$$x(s) = x(0) - \int_{\psi(0)}^{\psi(s)} \sin \eta R(\eta)\, d\eta,$$

and

$$y(s) = y(0) + \int_{\psi(0)}^{\psi(s)} \cos \eta R(\eta)\, d\eta,$$

since $R(\psi) = (ds/d\psi)$. We see that a convex curve can be recovered uniquely (up to translation) from its extended circular image.

Suppose now, that for convenience, we choose the starting point to be the point where $\psi = 0$. Then, integrating over the whole curve, we get

$$x(P) = x(0) - \int_0^{2\pi} \sin \psi R(\psi)\, d\psi,$$

and

$$y(P) = y(0) + \int_0^{2\pi} \cos \psi R(\psi)\, d\psi.$$

So a curve is closed if and only if

$$\int_0^{2\pi} \cos \psi R(\psi)\, d\psi = 0, \quad \text{and} \quad \int_0^{2\pi} \sin \psi R(\psi)\, d\psi = 0.$$

One way to interpret the above result is that the center of gravity of the circular mass distribution must be at the center of the unit circle, since

$$\int_0^{2\pi} R(\psi)(\cos \psi, \sin \psi)\, d\psi = (0, 0).$$

Equivalently, we see that the circular convolution of an extended circular image with the cosine function ($R(\psi) \otimes \cos \psi$) is identically equal to zero:

$$\int_0^{2\pi} R(\eta) \cos (\psi - \eta)\, d\eta = \cos \psi \int_0^{2\pi} R(\eta) \cos \eta\, d\eta + \sin \psi \int_0^{2\pi} R(\eta) \sin \eta\, d\eta,$$

which is 0 for all ψ. Thus $R(\psi)$ has no components at the fundamental frequency.

Convolutional filtering of the extended circular image $R(\psi)$ produces a closed curve, since linear operations cannot introduce new frequency components. Thus smoothed versions of a closed curve can be obtained by application of a smoothing filter to the extended circular image.

We can also conclude that any nonnegative function $R(\psi)$ that satisfies the closure condition corresponds uniquely (up to translation) to a convex curve defined by the equations for $x(s)$ and $y(s)$ given above.

VII. Convolution with a Filter

The circular convolution of an extended circular image $R(\psi)$ with a filter function $H(\psi)$ can be written as follows:

$$T(\psi) = \int_0^{2\pi} H(\psi - \eta)\, R(\eta)\, d\eta,$$

where it is assumed that H is periodic, so that its value for arguments outside the range $[0, 2\pi]$ can be found. The extended circular image, as here defined, is nonnegative. If the impulse response of the filter is nonnegative, that is, if it is a smoothing filter, the result will also be nonnegative. For other filters, the possibility exists that the result is negative in places, and so does not correspond to an extended circular image as here defined.

To show that the filtered result corresponds to a closed curve we need to show that the convolution of $T(\psi)$ with $\cos \psi$ is zero. Now convolution is associative and commutative, so

$$\begin{aligned} T \otimes \cos &= (R \otimes H) \otimes \cos \\ &= (H \otimes R) \otimes \cos = H \otimes (R \otimes \cos). \end{aligned}$$

The result is zero, since, as we showed above, $R(\psi) \otimes \cos \psi = 0$.

Sometimes it is more convenient to use a slightly different form of the filtering equation. Changing variables (from ψ to s) and noting that $R(\psi) = ds/d\psi$, we get

$$T(\psi) = \int_0^P H(\psi - \psi(s))\, ds,$$

or, equivalently,

$$\kappa(s) = 1 \Big/ \int_0^P H(\psi(s) - \psi(t))\, dt.$$

This form is particularly well suited for numerical calculations.

The two key differences between this and earlier methods for smoothing closed curves are:

- use of radius of curvature R instead of curvature, κ;
- use of the normal angle ψ instead of the arc-length s as independent variable.

As an example, consider convolving a square with a simple rectangular filter. The result is a figure composed of four circular arcs (see Fig. 1). We can see this by noting that the extended circular image of a square with sides of length L is composed of four impulses of weight L separated by 90°. Further, note that the area under the rectangular pulse has to be unity, in order for it to produce a figure with the same perimeter as the original curve. Thus, if the filter has angular extent θ, its amplitude must be $(1/\theta)$. So the result of the convolution will be four rectangular pulses of amplitude (L/θ) with angular extent θ. Each of these corresponds to a circular arc of radius (L/θ) and length

$$\frac{L}{\theta}\theta = L.$$

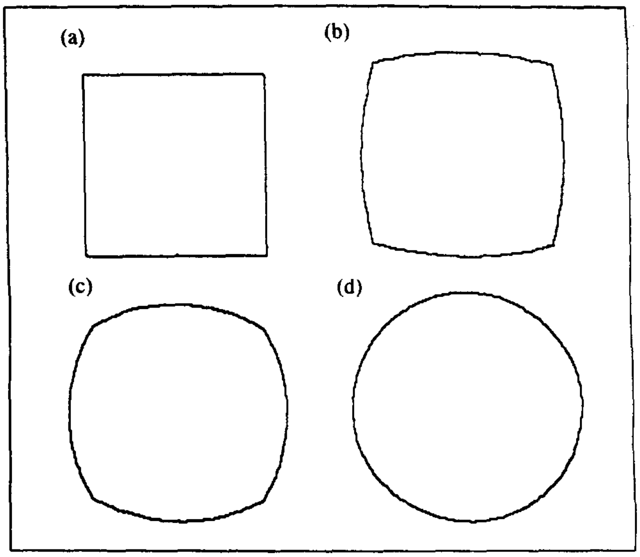

Fig. 1. Squares filtered with low-pass rectangular filters of various angular extents. (a) $\theta = 0°$. (b) $\theta = 30°$. (c) $\theta = 60°$. (d) $\theta = 90°$.

Shown in Fig. 1(b) and (c) are closed curves constructed by filtering the extended circular image of the square, shown in Fig. 1(a), with rectangular filters of angular extend 30° and 60°, respectively. If $\theta = \pi/2$, these arcs are tangent where they touch, and a circle is produced, as shown in Fig. 1(d). Finally, if $\theta > \pi/2$, the figure is actually composed of eight circular arcs, four with radius (L/θ) and four with radius $2(L/\theta)$. In this case there are no discontinuities in tangent direction.

VIII. Sum of Two Closed Curves

One can add the extended circular images of two convex curves. The result is the extended circular image of a new convex curve, with perimeter equal to the sum of the perimeters of the original curves. To see that the sum satisfies the closure condition, note that the sum of two functions with a zero fundamental frequency component is another function with a zero fundamental frequency component. Equivalently, the center of mass of the combination of two mass distributions lies along the line connecting the centers of mass of the two distributions. If both are at the center of the unit circle, the center of mass of their combination must also lie there.

The sum of the extended circular image of two convex polygons corresponds to the *mixed polygon* obtained from the two. Such objects find application in spatial reasoning [7]. They can also be computed by Boolean convolution of point sets in the plane.

An extended circular image can be scaled by multiplying it by some constant. The result corresponds to a closed figure of the same shape, but reduced or enlarged, depending on whether the constant is less than or greater than one.

One can consider all convex combinations of two extended circular images. If each is interpreted in terms of vectors to all points within the region that the curves enclose, then the result corresponds to the Minkowski sum [8]–[10]. The figures obtained by Boolean convolution are the same as the figures corresponding to the Minkowski sum.

As an example, consider the mixture of a square and a circle. The result is a figure composed of four circular arcs and four straight lines (see Fig. 2). We can see this by noting that the extended circular image of a square is composed of four impluses, while that of a circle is constant. In this particular case one can construct the result simply by cutting the circle into four quadrants and splicing these four sectors into the corners of the square. The exact shape of the result will depend on the relative sizes of the original figures.

We introduce the concept of the mixed figure here, because it provides us with another way of thinking about the smoothing operation described earlier. One may consider the convolution integral as the limit of a sum of products of values of one function with values of a shifted version of the other. That is,

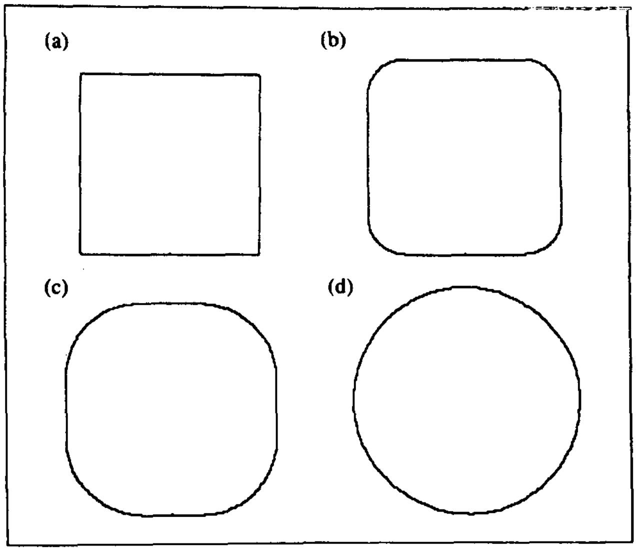

Fig. 2. Combinations of squares and circles. (a) Square. (b) $\frac{2}{3}$ square, $\frac{1}{3}$ circle. (c) $\frac{1}{3}$ square, $\frac{2}{3}$ circle. (d) Circle.

$$\int_0^{2\pi} H(\eta)\, R(\psi - \eta)\, d\eta = \lim_{n\to\infty} \sum_{i=0}^{n} H\left(i\frac{2\pi}{n}\right) R\left(\psi - i\frac{2\pi}{n}\right)\frac{2\pi}{n}.$$

Since addition of extended circular images corresponds to mixing of figures, one can regard the filtered version of a curve as the limit of the mixture of many scaled and rotated copies of the figure. This idea is sometimes helpful in visualizing the effects of particular filtering operations on given closed figures.

IX. Integral of the Extended Circular Image

Note that $R(\psi)$ is periodic, as is $\kappa(s)$. Their integrals, $s(\psi)$ and $\psi(s)$, however, are not. Filtering operations on $R(\psi)$ and $\kappa(s)$ do not, in general, have the same effect as similar filtering operations on $s(\psi)$ and $\psi(s)$. This may seem surprising since differentiation is a linear process and thus should commute with convolution. The problem is that we are dealing with circular convolution which applies to periodic functions only. When integrals involving $s(\psi)$, for example, are expanded using integration by parts, one obtains not only the appropriate integral containing $R(\psi)$, but an additional term which is nonzero because $s(\psi)$ is not periodic.

When necessary, one can define the following periodic functions

$$s'(\psi) = s(\psi) - \frac{\psi}{2\pi} P \quad \text{and} \quad \psi'(s) = \psi(s) - \frac{s}{P}(2\pi),$$

whose derivatives are

$$R'(\psi) = R(\psi) - \frac{P}{2\pi} \quad \text{and} \quad \kappa'(s) = \kappa(s) - \frac{2\pi}{P}.$$

Filtering operations on $R'(\psi)$ are equivalent to filtering operations on $s'(\psi)$. This is a useful relationship, since it is sometimes easier to see what is happening by looking at the $s(\psi)$ curve than directly considering the extended circular image $R(\psi)$. The integrals of the products of $s'(\psi)$ and $R'(\psi)$ with cos ψ and sin ψ are zero. Note however that $R'(\psi)$ does not correspond to an extended circular image, as defined here, since it will be negative in places.

X. Discrete Approximation

To represent an extended circular image in a computer one may choose to discretize directions and collect a *direction histogram.* The direction histogram is the two dimensional analog of the orientation histogram used to represent extended Gaussian images [6]. It shows for each angular interval how much of the curve has tangent directions falling into that angular interval. An approximation can be calculated easily by dividing the curve into many short segments. One simply determines the length and tangent direction of each segment and adds the length to the histogram bin corresponding to that tangent direction. This shows clearly that the calculation of the extended circular image $R(\psi)$ does not require taking higher derivatives, as does the computation of the curvature $\kappa(s)$ from $\psi(s)$.

The discrete approximation also suggests a possible extension to nonconvex curves:

$$R(\psi) = \sum_{i=1}^{n} \frac{1}{|\kappa(s_i)|},$$

where s_i identifies the ith place on the curve where the tangent equals ψ. The result will be an extended circular image that satisfies the condition for closure given above. As a result it corresponds uniquely to a particular convex curve.

Note that we do not allow the contribution from a concave part of the curve to be negative. Otherwise, this natural extension would cause the closure condition to be violated. It also would require higher derivatives in the computation of the extended circular image.

We obtain the extension to nonconvex curves, shown above, by considering the integrals for $x(s)$ and $y(s)$ first broken up into n integrals over segments where curvature has constant sign, before changing variables. Suppose that n zero crossings of curvature occur at $s = s_i$, for $i = 1$ to n. Then, the convenience, let $s_0 = 0$ and $s_{n+1} = P$.

$$\int_0^P \cos\psi(s)\, ds = \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} \cos\psi(s)\, ds.$$

So, upon changing variables,

$$\int_0^P \cos\psi(s)\, ds = \sum_{i=0}^{n} \int_{\psi(s_i)}^{\psi(s_{i+1})} \cos\psi G_i(\psi)\, d\psi.$$

Then

$$\int_0^P \cos\psi(s)\, ds = \int_0^{2\pi} \cos\psi G(\psi)\, d\psi,$$

where

$$G(\psi) = \sum_{i=0}^{n} |G_i(\psi)|,$$

and $G_i(\psi) = ds/d\psi$ in the ith segment. (We have to take the absolute value in the sum to account for the fact that we are integrating from larger angles to smaller angles in those segments of the curve where $G(\psi)$ is negative.)

XI. Conclusion

We have shown that a novel representation of a simple, convex, closed curve makes it possible to obtained smoothed, closed versions by a simple convolution operation. We have yet to determine whether this approach is applicable to nonconvex curves.

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