KINEMATICS, STATICS, AND DYNAMICS OF TWO-DIMENSIONAL MANIPULATORS

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In order to get some feeling for the kinematics, statics, and dynamics of manipulators, it is useful to separate visualization of linkages in three-space from basic mechanics. The general-purpose two-dimensional manipulator is analyzed in this paper in order to gain a basic understanding of the mechanics issues without encumbrance from the complications of three-dimensional geometry.

Introduction

Kinematics deals with the basic geometry of the linkages. If we consider an articulated manipulator as a device for generating position and orientation, we need to know the relationships between these quantities and the joint variables, since it is the latter that we can easily measure and control. Position here refers to the position in space of the tip of the device, while orientation refers to the direction of approach of the last link. While position is fairly easy to understand in spaces of higher dimensionality, rotation or orientation rapidly becomes more complex. This is the main impetus for our study of two-dimensional devices. In two dimensions, two degrees of freedom are required to generate arbitrary positions in a given work space and one more is needed to control the orientation of the last link.

The device studied in detail has only two joints and so can be used as a position generator. A three-link device is a general-purpose two-dimensional device that can generate orientation as well.

It will become apparent that the calculation of position and orientation of the last link given the joint variables is straightforward, while the inverse calculation is hard and may be intractable for devices with many links that have not been designed properly. The calculation of joint angles given desired position and orientation is vital if around or follow a given trajectory.

If a manipulator has just enough degrees of freedom to cover its work space, there will in general be a finite number of ways of reaching a given position and orientation. This is because the inverse problem essentially corresponds to solving a number of equations in an equal number of unknowns. If the equations were linear we would expect exactly one solution. Since they are trigonometric polynomials in the joint variables -- and hence nonlinear -- we expect a finite number of solutions. Similarly, if we have too few joints, there will in general be no solution, while with too many joints we expect an infinite number of ways of reaching a given position and orientation. Usually there are some arm configurations that present special problems because the equations become singular. These often occur on the boundary of the work space, where some of the links become parallel.

Statics deals with the balance of forces and torques required when the device does not move. If we consider an articulated manipulator as a device for applying forces and torques to objects being manipulated, we need to know the relationship between these quantities and the joint torques, since it is the latter that we either directly control or can at least measure. In two dimensions, two degrees of freedom will be required to apply an arbitrary force at the tip of the device and one more if we want to control torque applied to the object as well.

Clearly then the two-link device to be discussed can be thought of as a force generator, while the three-link device can apply controlled torques as well. The gravity loading of the links has to be compensated for as well and fortunately it can be considered separately from the torques required to produce tip forces and torques.

Dynamics deals with the manipulator in motion. It will be seen that the joint torques control the angular accelerations. The relationships are not direct however. First of all, the sensitivity of a given joint to torque varies with the arm configuration; secondly, forces appear that are functions of the products of the angular velocities; and thirdly there is considerable coupling between the motions of the links. The velocity product terms can be thought of as generalized centrifugal forces.

The equations relating joint accelerations to joint torques are nonlinear, but given the arm state -- that is both joint variables and their rate of change with time -- it is straightforward to calculate what joint torques are required to achieve given angular accelerations. We can, in other words, calculate the time-history of motor torques for each joint required to cause the arm to follow a given trajectory.

Notice that this is an open-loop dead-reckoning approach which in practice has to be modified to take into account friction and small errors in estimating the numerical constant in the sensitivity matrix. The modification can take the form of a small amount of compensating feedback.

This, however, should not be confused with the more traditional, analog servo methods which position-controls each joint independently. Since the dynamic state of the manipulator is a global property, one cannot expect general success using local, joint independent position-control.

To summarize: we will deal with unconstrained motion of the manipulator as it follows some trajectory as well as its interaction with parts that mechanically constrain its motion. Both aspects of manipulator operation are of importance if it is to be used to assemble or disassemble artifacts.

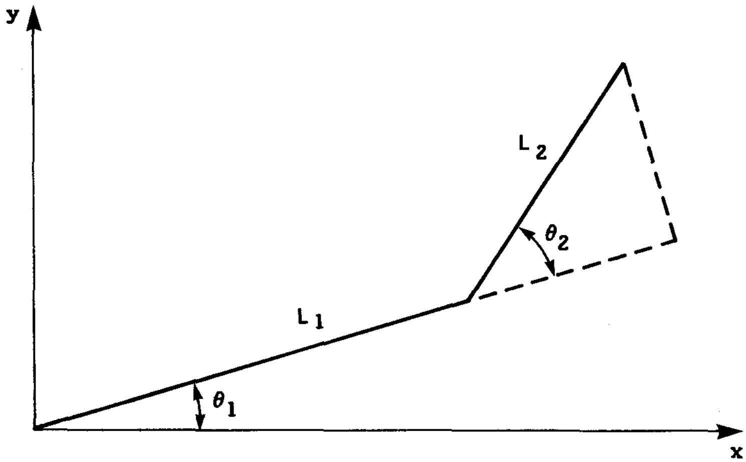
Two-link Manipulator Kinematics

In two dimensions one clearly needs two degrees of freedom to reach an arbitrary point within a given work space. Let us first study a simple two-link manipulator with rotational joints. Note that the geometry of the two-link device occurs as a subproblem in many of the more complicated manipulators. Given the two joint angles, let us calculate the position of the tip of the device. Define vectors corresponding to the two links:

$$\underline{r}_1 = l_1\,[\cos(\theta_1), \sin(\theta_1)]$$
$$\underline{r}_2 = l_2\,[\cos(\theta_1 + \theta_2), \sin(\theta_1 + \theta_2)]$$

The the position of the tip $\underline{r}$ can be found simply by vector addition.

$$x = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$



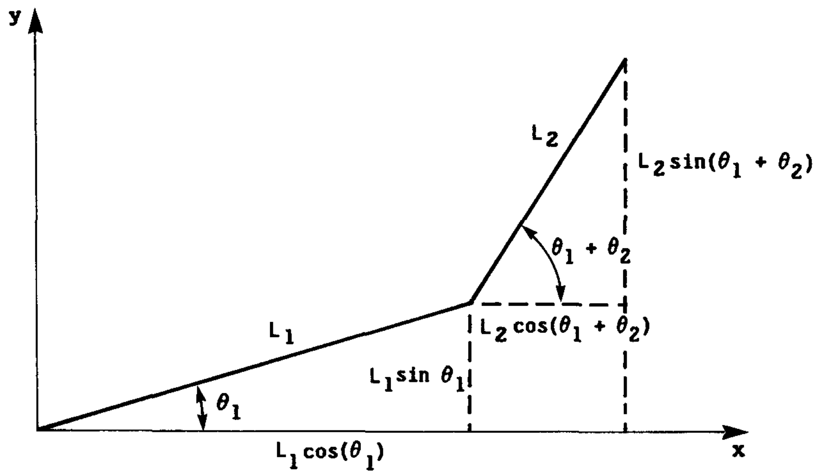
This can be expanded into a slightly more useful form:

$$x = [l_1 + l_2 \cos(\theta_2)] \cos(\theta_1) - l_2 \sin(\theta_2) \sin(\theta_1)$$
$$y = [l_1 + l_2 \cos(\theta_2)] \sin(\theta_1) - l_2 \sin(\theta_2) \cos(\theta_1)$$

The Inverse Problem

While the forward calculation of tip position from joint angles is always relatively straightforward, the inversion is intractable for manipulators with more than a few links unless the device has been specially designed with this problem in mind. For our simple device we easily get:

$$\cos(\theta_2) = \frac{(x^2 + y^2) - (l_1^2 + l_2^2)}{2\ l_1\ l_2}$$


y
x
L_1
L_2
θ_1
$\theta_1 + \theta_2$
$L_1 \cos(\theta_1)$
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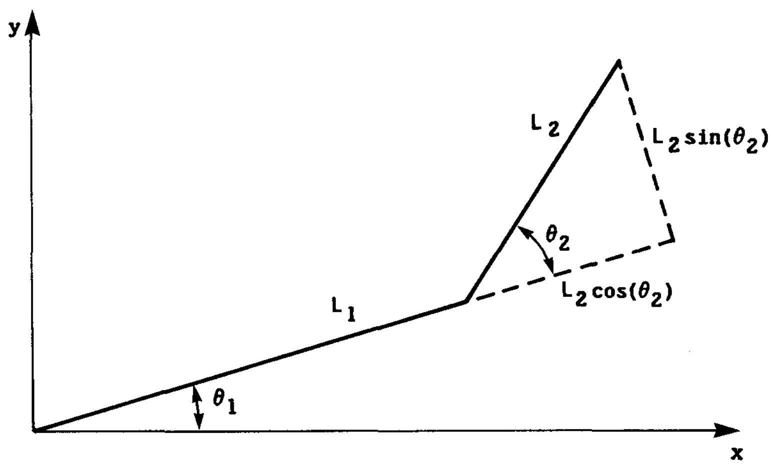
y
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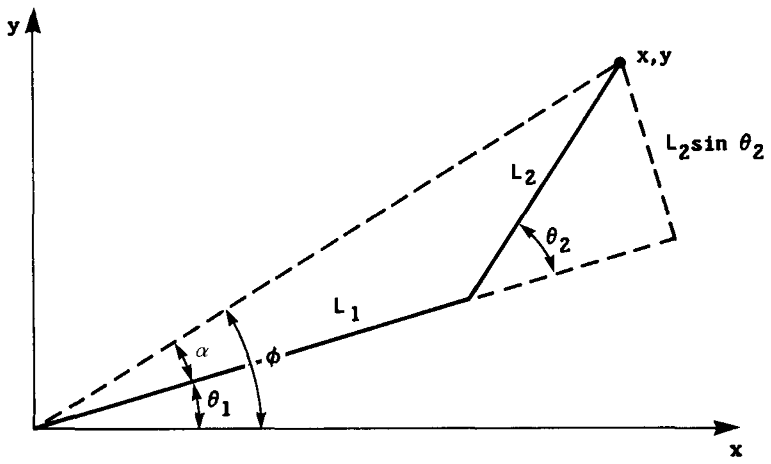
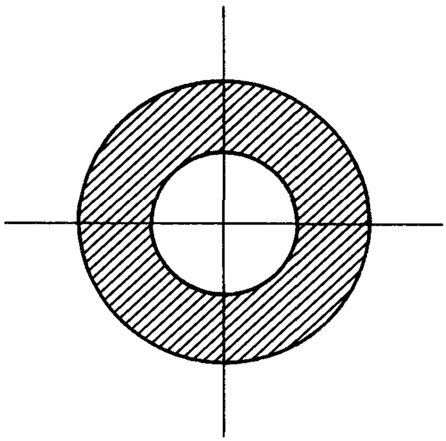


There will be two solutions for θ_2 of equal magnitude and opposite sign. Expanding $\tan(\theta_1) = \tan(\theta - \alpha)$ and using $\tan(\theta) = y/x$ we also arrive at:

$$\tan(\theta_1) = \frac{-l_2 \sin(\theta_2)\, x + [l_1 + l_2 \cos(\theta_2)]y}{l_2 \sin(\theta_2)\, y + [l_1 + l_2 \cos(\theta_2)]x}$$

The reason this was so easy is that we happened to have already derived all the most useful formula using geometric and trigonometric reasoning. A method of more general utility depends on algebraic manipulation of the expressions for the coordinates of the tip. Notice that these expressions are polynomials in the sines and cosines of the joint angles. Such systems of polynomials can be solved systematically -- unfortunately the degree of the intermediate terms grows explosively as more and more variables are eliminated. So this method, while quite general, is in practice limited to solving only simple linkages.

The Work Space



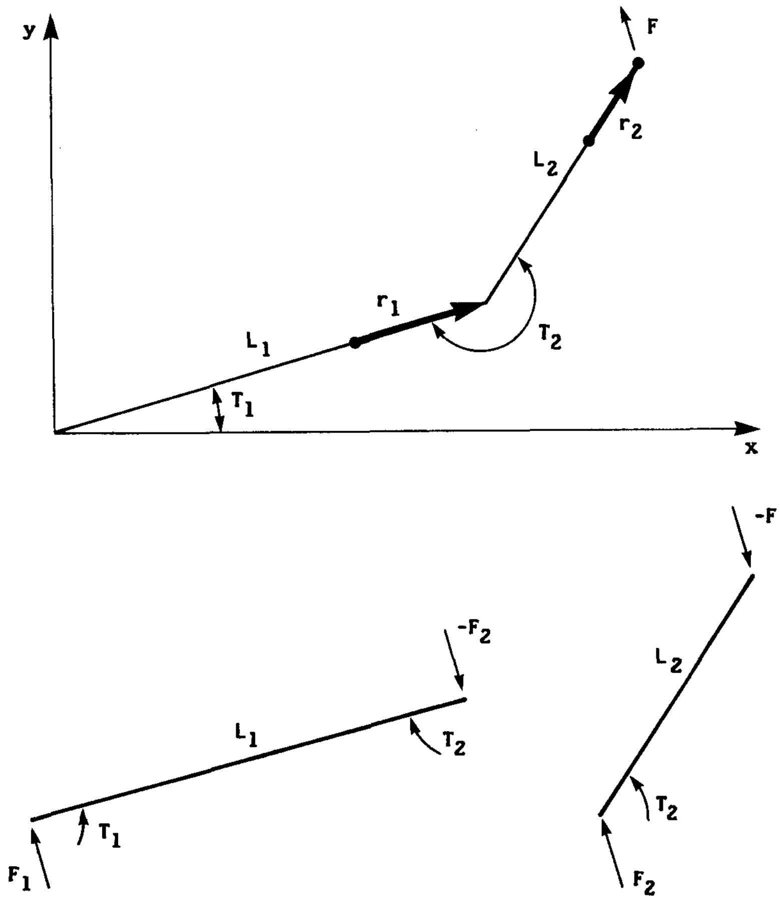
$$(l_1 - l_2)^2 \le l_1^2 + 2\, l_1\, l_2 \cos(\theta_2) + l_2^2 \le (l_1 + l_2)^2$$

So:

$$|l_1 - l_2| \le \sqrt{x^2 + y^2} \le |l_1 + l_2|$$

The set of points reachable by the tip of the device is an annulus centered on the origin. Notice that points on the boundary of this region can be reached in one way, while points inside can be reached in two. The width of the annulus is twice the length of the shorter link and its average radius equals the length of the longer one.

When $l_1 = l_2 = 1$ say, the work space becomes simpler, just a circle. The origin is a singular point in that it can be reached in an infinite number of ways since θ_1 can be chosen



freely.

Statics

So far we have thought of the manipulator as a device for placing the tip in any desired position within the work space -- that is, a position generator. Equally important is the device's ability to exert forces on objects. Let us assume that the manipulator does not move appreciably when used in this way so that we can ignore torques and forces used to accelerate the links. Initially we will also ignore gravity; we will later calculate the additional torques required to balance gravity components.

We have direct control over the torques T_1 and T_2 generated by the motors driving the joints. What forces are produced by these torques at the tip? Since we do not want the device to move, imagine its tip pinned in place. Let the force exerted by the tip on the pin be $\underline{F} = (u,v)$. To find the relationships between the forces at the tip and the motor torques, we will write down one equation for balance of forces and one equation for balance of torques for each of the links. Writing down the equations for balance of forces in each of the two links we get:

$$\underline{F}_1 = \underline{F}_2 \quad \text{and} \quad \underline{F}_2 = \underline{F}, \quad \text{that is } \underline{F} = \underline{F}_1 = \underline{F}_2$$

Next picking an arbitrary axis for each of the links we get the equations for balance of torques:

$$\begin{aligned} T_1 - T_2 &= \underline{r}_1 \times \underline{F} \\ T_2 &= \underline{r}_2 \times \underline{F} \end{aligned}$$

Where [a,b] x [c,d] = ad-bc is the vector cross-product.

$$\begin{aligned} T_1 &= \underline{r}_1 \times \underline{F} + T_2 \\ &= \underline{r}_1 \times \underline{F} + \underline{r}_2 \times \underline{F} \\ &= (\underline{r}_1 + \underline{r}_2) \times \underline{F} \end{aligned}$$

If $T_2 = 0$, then $\underline{r}_2 \times \underline{F} = 0$ and so $\underline{r}_2$ and $\underline{F}$ must be parallel, while $T_1 = 0$, gives $(\underline{r}_1 + \underline{r}_2) \times \underline{F} = 0$ and $(\underline{r}_1 + \underline{r}_2)$ is parallel to $\underline{F}$. These directions for $\underline{F}$ are counter-intuitive if anything!

Expanding the cross-products we get:

$$T_1 = [l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)]\, v - [l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)]\, u$$

$$T_2 = [l_2 \cos(\theta_1 + \theta_2)]\, v - [l_2 \sin(\theta_1 + \theta_2)]\, u$$

Using these results we can easily calculate what torques the motors should apply at the joints to produce a desired force at the tip.

The Inverse Statics Problem

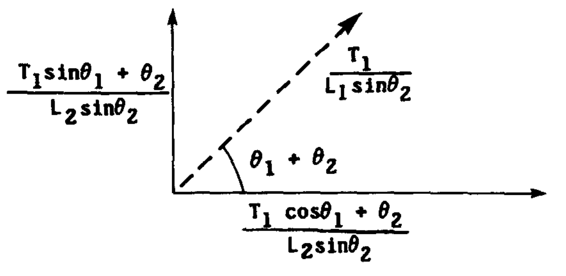
Now suppose we want to invert this process to calculate the force at the tip given measured joint torques. Fortunately this inversion is straightforward; we simply solve the pair of equations for u and v:

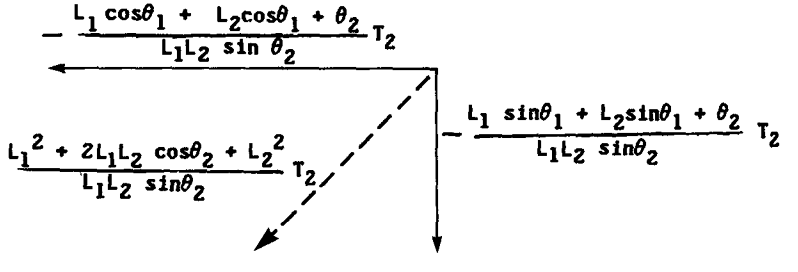
$$u = \frac{\{l_2 \cos(\theta_1 + \theta_2)\, T_1 - [l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)]\, T_2\}}{[l_1\, l_2 \sin(\theta_2)]}$$

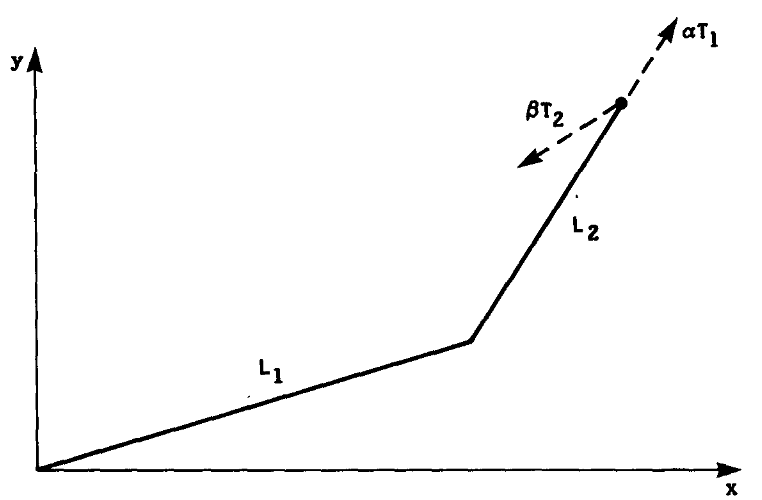
$$v = \frac{\{l_2 \sin(\theta_1 + \theta_2)\, T_1 - [l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)]\, T_2\}}{[l_1\, l_2 \sin(\theta_2)]}$$

Now we can see in quantitative terms the force components produced by each joint torque acting on its own:

There are singularities in the transformation when $\sin(\theta_2) = 0$, that is when $\theta_2 = 0$ or π. Obviously when the links are parallel, the joint torques have no control over the force component along the length of the links. Again we see the special nature of the boundary of the work space.

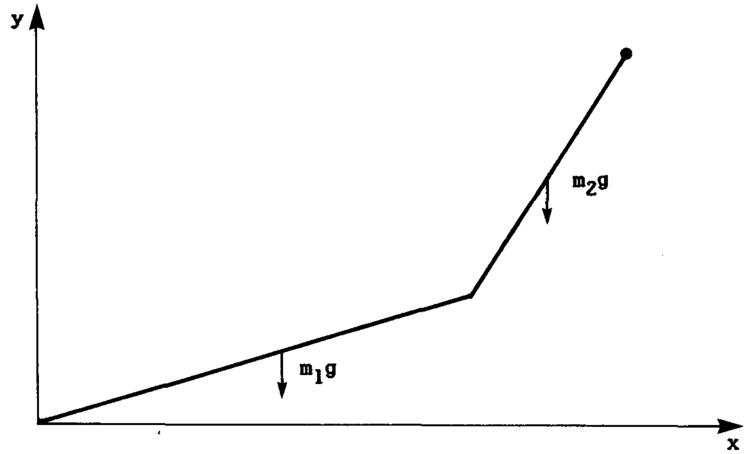

$\frac{T_1 \sin\theta_1 + \theta_2}{L_2 \sin\theta_2}$
$\frac{T_1}{L_1 \sin\theta_2}$
$\theta_1 + \theta_2$
$\frac{T_1 \cos\theta_1 + \theta_2}{L_2 \sin\theta_2}$


$-\frac{L_1 \cos\theta_1 + L_2 \cos\theta_1 + \theta_2}{L_1 L_2 \sin\theta_2} T_2$
$-\frac{L_1 \sin\theta_1 + L_2 \sin\theta_1 + \theta_2}{L_1 L_2 \sin\theta_2} T_2$
$\frac{L_1^2 + 2L_1L_2 \cos\theta_2 + L_2^2}{L_1 L_2 \sin\theta_2} T_2$


y
x
L_1
L_2
αT_1
βT_2

Balancing Gravity

Let us assume for concreteness that the center of mass of each link is at its geometric center and let us define a gravity vector $\underline{g} = [0,-g]$ acting in the negative y direction. We could now repeat the above calculation with two additional components in the force-balance equations due to the gravity loading. Inspection of the equations shows that the resultant torques are linear in the applied forces, so we can use the principle of superposition, and calculate the gravity induced torques separately. Where there is no applied force at the tip we find



that $\underline{F}_2 = m_2\underline{g}$ and

$$\underline{F}_1 = \underline{F}_2 + m_1\underline{g} = (m_1 + m_2)\underline{g}$$

Considering the torques we find:

$$T_{2g} = -m_2(1/2)\underline{r}_2 \times \underline{g} = g[(1/2)m_2\, l_2 \cos(\theta_1 + \theta_2)]$$
$$T_{1g} = T_{2g} - m_1\, 1/2\, \underline{r}_1 \times \underline{g} - m_2\, \underline{r}_1 \times \underline{g}$$
$$= g[(1/2)m_1 + m_2)\, l_1 \cos(\theta_1) + (1/2)m_2 l_2 \cos(\theta_1 + \theta_2)]$$

These terms can now be added to the torque terms derived earlier for balancing the force applied at the tip.

Dynamics

Now let us determine what happens if we remove the pin holding the manipulator tip in place and then apply torques to the joints. What angular accelerations of the links will be produced? Knowing the relation between these two quantities will allow us to control the motions of the device as it follows some desired trajectory. We could proceed along lines similar to the ones followed when we studied statics, simply adding Newton's law.

$$\underline{F} = m\underline{a} \qquad \text{or} \qquad T = I\alpha$$

where $\underline{F}$ is a force, m mass and $\underline{a}$ linear acceleration. Similarly T is a torque, I moment of inertia and α angular acceleration. The quantities involved would have to be expressed relative to some Cartesian coordinate system. We would be faced with large sets of nonlinear equations, since the mechanical constraints introduced by the linkage would have to be explicitly included and the coordinates of each joint expressed. In general, this method becomes quite unwieldly for manipulators with more than a few links. The more general form of Newton's law indicates a better approach:

$$F_i = d/dt\ (mv_i)$$

where F_i is a component of the force and mv_i is a component of the linear momentum. It is possible to develop a similar equation in a generalized coordinate system that does not have to be Cartesian. It is natural to chose the joint angles as the generalized coordinates. These provide a compact description of the arm configuration and the mechanical constraints are implicitly taken care of. It can be shown that:

$$Q_i = d/dt\ p_i - \partial L/\partial q_i$$

where Q_i is a generalized force, p_i generalized momentum and q_i one of the generalized coordinates. There is one such equation for each degree of freedom. Q_i will be a force for an extensional joint, and a torque for a rotational joint. In both cases, $Q_i q_i$ has the dimensions of work.

Dynamics using Lagranges Equation

In this relation, L is the Lagrangian or "kinetic potential," equal to the difference between kinetic and potential energy, K - P. The generalized momentum p_i can be expressed in terms of L:

$$p_i = \partial L/\partial \dot{q}_i$$

This is analogous to

$$mv = d/dv\ (1/2\ mv^2).$$

The dot represents differentiation with respect to time. Finally:

$$d/dt\ (\partial L/\partial \dot{q}_i) - \partial L/\partial q_i = Q_i$$

Once again there is one such equation for each degree of freedom of the device.

It will be convenient to ignore gravity on the first round -- so there will be no potential energy term. Next we will take the simple case of equal links and let the links be sticks of equal mass m and uniform mass distribution. The moment of inertia for rotation about the center of mass of such a stick is $(1/12)ml^2$. These assumptions allow a great deal of simplification of intermediate terms without losing much of importance. In fact the final result would be the same, except for some numerical constants if we had considered the more general case.

Kinetic energy of a rigid body can be decomposed into a component due to the instantaneous linear translation of its center of mass (1/2 mv^2) and a component due to the instantaneous angular velocity (1/2 $I\omega^2$). The angular velocities obviously are just $\dot{\theta}_1$ and $(\dot{\theta}_1 + \dot{\theta}_2)$. The magnitudes of the instantaneous linear velocities of the center of mass are:

$$|1/2\ \underline{r}_1\ \dot{\theta}_1|$$
$$|\underline{r}_1\dot{\theta}_1 + 1/2\ \underline{r}_2(\dot{\theta}_1 + \dot{\theta}_2)|$$

The squares of these quantities are:

$$(1/4)l^2\ \dot{\theta}_1^{\,2} \qquad \text{and}$$
$$l^2[\dot{\theta}_1^{\,2} + \cos(\theta_2)\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) + 1/4(\dot{\theta}_1 + \dot{\theta}_2)^2]$$

The total kinetic energy of link 1 is then:

$$1/2(1/12)\ ml^2\dot{\theta}_1^{\,2} + (1/2)m(1/4)l^2\ \dot{\theta}_1^{\,2} = 1/2((1/3)ml^2)\ \dot{\theta}_1^{\,2}$$

The same result could have been obtained more directly by noting that the moment of inertia of a stick about one of its ends is $(1/3)ml^2$.

The total kinetic energy of link 2 is:

$$\begin{aligned} &1/2((1/12)ml^2)(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ &\quad + (1/2)ml^2[(5/4 + \cos(\theta_2))\dot{\theta}_1^{\,2} \\ &\qquad\qquad + (1/2 + \cos(\theta_2))\dot{\theta}_1\dot{\theta}_2 + (1/4)\dot{\theta}_2^{\,2}] \\ &= (1/2)ml^2\ [(4/3 + \cos(\theta_2))\ \dot{\theta}_1^{\,2} + (2/3 + \cos(\theta_2))\ \dot{\theta}_1\dot{\theta}_2 \\ &\qquad + 1/3\ \dot{\theta}_2^{\,2}] \end{aligned}$$

Finally, adding all components of the kinetic energy and noting the P = 0, we determine the Lagrangian:

$$L = (1/2)ml^2 \, [(5/3 + \cos(\theta_2)) \, \dot\theta_1^{\,2} + (2/3 + \cos(\theta_2)) \, \dot\theta_1\dot\theta_2 + 1/3 \, \dot\theta_2^{\,2}]$$

Next we will need the partial derivative of L with respect to θ_1, θ_2, $\dot\theta_1$, $\dot\theta_2$. For convenience let $L' = L/(1/2)ml^2$.

$$\begin{aligned}
\partial L'/\partial\theta_1 &= 0 \\
\partial L'/\partial\theta_2 &= -\sin(\theta_2) \, \dot\theta_1(\dot\theta_1 + \dot\theta_2) \\
\partial L'/\partial\dot\theta_1 &= 2(5/3 + \cos(\theta_2))\dot\theta_1 + (2/3 + \cos(\theta_2))\dot\theta_2 \\
\partial L'/\partial\dot\theta_2 &= (2/3 + \cos(\theta_2)) \, \dot\theta_1 + 2 \, (1/3) \, \dot\theta_2
\end{aligned}$$

We will also require the time derivatives of these last two expressions:

$$\begin{aligned}
&d/dt \, (\partial L'/\partial\dot\theta_1) \\
&\quad = \ddot\theta_1 \, 2 \, (5/3 + \cos(\theta_2)) + \ddot\theta_2(2/3 + \cos(\theta_2)) \\
&\qquad - \sin(\theta_2)\dot\theta_2(2\dot\theta_1 + \dot\theta_2) \\
&d/dt \, (\partial L'/\partial\dot\theta_2) \\
&\quad = \ddot\theta_1 \, (2/3 + \cos(\theta_2)) + \ddot\theta_2 \, 2(1/3) \\
&\qquad - \sin(\theta_2) \, \dot\theta_1\dot\theta_2
\end{aligned}$$

When we plug all this into Lagrange's equation

$$d/dt \, (\partial L/\partial\dot\theta_i) - \partial L/\partial\theta_i = T$$

we get:

$$\begin{aligned}
&\ddot\theta_1 \, 2(5/3 + \cos(\theta_2)) + \ddot\theta_2 \, (2/3 + \cos(\theta_2)) \\
&\qquad = T_1/(1/2)ml^2 + \sin(\theta_2 \,) \, \dot\theta_2(2\dot\theta_1 + \dot\theta_2) \\
&\ddot\theta_1 \, (2/3 + \cos(\theta_2)) + \ddot\theta_2 \, 2(1/3) \\
&\qquad = T_2/(1/2)ml^2 - \sin(\theta_2) \, \dot\theta_1^{\,2}
\end{aligned}$$

And if you think that was painful, try it the other way! So finally we have a set of equations that allow us to calculate joint torques given desired joint accelerations. Notice that we need to know the arm state, θ_1, θ_2, $\dot\theta_1$, and $\dot\theta_2$ in order to do this. In part this is because of the appearance of velocity-product terms, representing centrifugal forces and the like, and in part it is because the coefficients of the accelerations vary with the arm configuration. It is useful to separate out these latter terms which constitute the sensitivity matrix.

$$\begin{bmatrix} 2(5/3 + \cos(\theta_2)) & (2/3 + \cos(\theta_2)) \\ (2/3 + \cos(\theta_2)) & 2(1/3) \end{bmatrix}$$

If we ignore the velocity-product terms, this matrix tells us the sensitivity of the angular accelerations with respect to the applied torques. It can be shown that the terms in this matrix will depend only on the generalized coordinates (and not the velocities), that the matrix must be symmetrical and that the diagonal terms must be positive.

This, by the way, implies that if one makes the torques large enough to overcome the velocity-product terms, the links will move in the expected direction. The analog, positional approach to arm control depends critically on this property. Notice the couplings between links -- that is torque applied to one joint will cause angular accelerations of both links in general.

The Inverse Matrix

If we wish to know exactly what accelerations will be produced by given torques we have to solve for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ in the above equations.

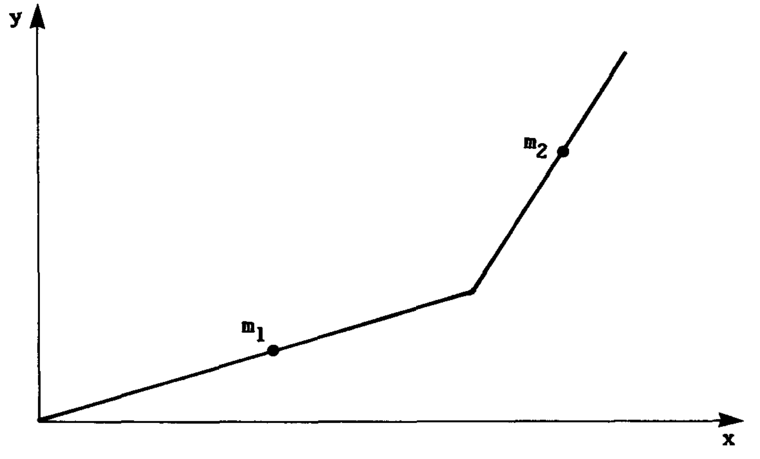
$$\ddot{\theta}_1 = [2(1/3)\ T_1' - (2/3 + \cos(\theta_2))\ T_2'] \ /\ (16/9 - \cos^2(\theta_2))$$

$$\ddot{\theta}_2 = \frac{[-(2/3 + \cos(\theta_2))\ T_1' + 2(5/3 + \cos(\theta_2))T_2']}{(16/9 - \cos^2(\theta_2))}$$

where

$$T_1' = T_1/(1/2\ ml^2) + \sin(\theta_2)\dot{\theta}_2(2\dot{\theta}_1 + \dot{\theta}_2)$$
$$T_2' = T_2/(1/2\ ml^2) - \sin(\theta_2)\ \dot{\theta}_1^{\,2}$$

Taking Gravity into Acount



We can define the potential energy P as the sum of the products of the link masses and the elevation of their center of mass relative to some arbitrary place.

$$P = \{gm_1\ (1/2)l_1\ \sin(\theta_1)\}$$
$$+ \{gm_2[l_1\ \sin(\theta_1) + (1/2)l_2\ \sin(\theta_1 + \theta_2)]\}$$

We could now repeat the above calculation, subtracting this term from the kinetic energy. Because of the linearity of the equations, we can again make use of superposition and calculate the torques required to balance gravity separately. Now the partial derivative of P with respect to the angular velocities are 0 so we only need the following:

$$T_{1g} = \partial P/\partial\theta_1 = g[(1/2\ m_1 + m_2)l_1 \cos(\theta_1) + 1/2\ m_2 l_2 \cos(\theta_1 + \theta_2)]$$
$$T_{2g} = \partial P/\partial\theta_2 = g[1/2\ m_2\ l_2 \cos(\theta_1 + \theta_2)]$$

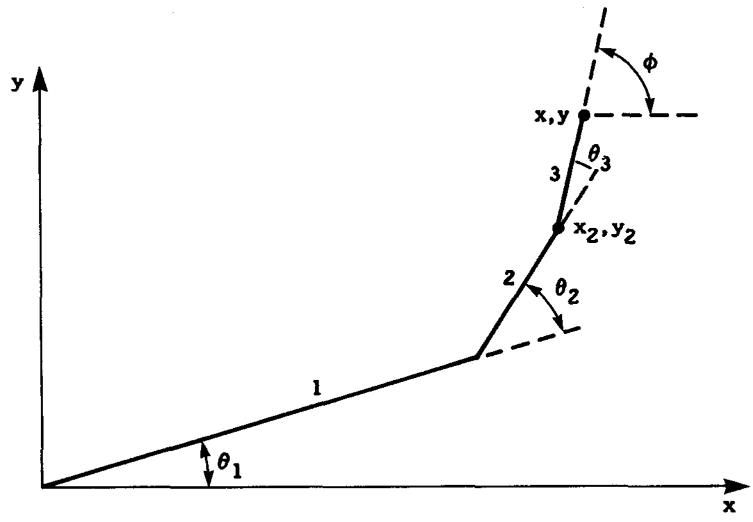
Adding a Third Link

A manipulator not only has to be able to reach points within a given work space, it also has to be able to approach the object to be manipulated with various orientations of the terminal device. That is, we need a position and orientation generator. Similarly it can be argued that it should not only be able to apply forces to the object, but torques as well. Additional degrees of freedom are required to accomplish this. If we are confined to operation in a two-dimensional space only one extra degree of freedom will be needed, since rotation can take place only about one axis, the axis normal to the plane of operation. It turns out that the same can be said about torque, since applying a torque can be thought of as an attempt to cause a rotation. So in two dimensions, a three link manipulator is sufficient for our purposes. We will now repeat our analysis of kinematics, statics, and dynamics for this device, but with fewer details than before.

Kinematics with Three Links

$$x = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos\ (\theta_1 + {}_2 + \theta_3)$$
$$y = l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3)$$
$$\phi = \theta_1 + \theta_2 + \theta_3$$

While we could proceed to solve the inverse problem of finding joint angles from tip position and orientation by geometric, trigonometric or algebraic methods, it is simpler to make use of the results for the two-link manipulator. One can easily calculate



the position of joint 2, knowing

$$x_2 = x - l_3 \cos(\phi)$$
$$y_2 = y - l_3 \sin(\phi)$$

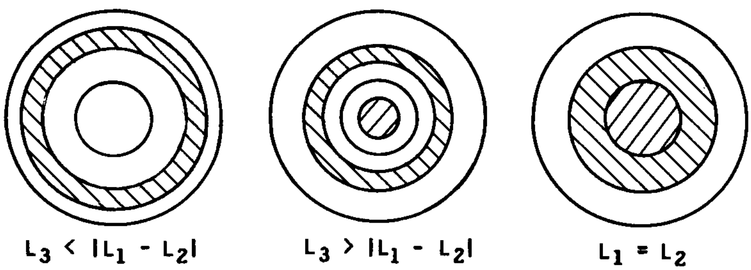
Now one can simply solve the remaining two-link device precisely as before:

$$\cos(\theta_2) = [(x_2^2 + y_2^2) - (l_1^2 + l_2^2)] / 2\, l_1\, l_2$$
$$\tan(\theta) = y_2/x_2$$
$$\tan(\alpha) = l_2 \sin(\theta_2) / [l_1 + l_2 \cos(\theta_2)]$$
$$= 2\, l_1\, l_2 \sin(\theta_2) / [(x_2^2 + y_2^2) + (l_1^2) + (l_1^2 - l_2^2)]$$
$$\theta_1 = \theta - \alpha$$
$$\theta_3 = \phi - \theta_2 - \theta_1$$

To determine how much of the work space that can be reached by the manipulator is usable with arbitrary orientation of the last link, we could, as before, proceed with an algebraic approach.

For example we might start from $|\cos(\theta_2)| \leq 1$ and the realization that the worst case situations occur when the last link is parallel to the direction from the origin to the tip. The situation is easy enough to visualize, so we will use geometric reasoning instead.

Not all points in the annular work space previously determined can be reached with arbitrary orientation of the last link. A method for constructing the usable work space is simply to construct a circle of radius l_3 about each point. A point is in the usable work space if the circle so constructed lies inside the annulus previously determined.



Statics with Three Links

We have control of over the three torques T_1, T_2, and T_3. We would like to use these to apply force $\underline{F} = (u,v)$ and torque T to the object held by the tip of the device. We do not want to consider motion of the manipulator now, so again imagine its tip solidly fixed in place. We proceed by writing down one equation for force balance for each link and one equation for torque balance for each link.

$$\underline{F}_1 = \underline{F}_2,\ \underline{F}_2 = \underline{F}_3 \text{ and } \underline{F}_3 = \underline{F} \text{ so } \underline{F} = \underline{F}_1 = \underline{F}_2 = \underline{F}_3$$
$$T_3 = T + \underline{r}_3 \times \underline{F}$$
$$T_2 = T_3 + \underline{r}_2 \times \underline{F}_3$$
$$T_1 = T_2 + \underline{r}_1 \times \underline{F}_2$$
$$T_1 = T + (\underline{r}_1 + \underline{r}_2 + \underline{r}_3) \times \underline{F}$$
$$T_2 = T + (\underline{r}_2 + \underline{r}_3) \times \underline{F}$$
$$T_3 = T + (\underline{r}_3) \times \underline{F}$$

Let's abbreviate the trigonometric terms by subscripts on the letters "s" and "c", so for example

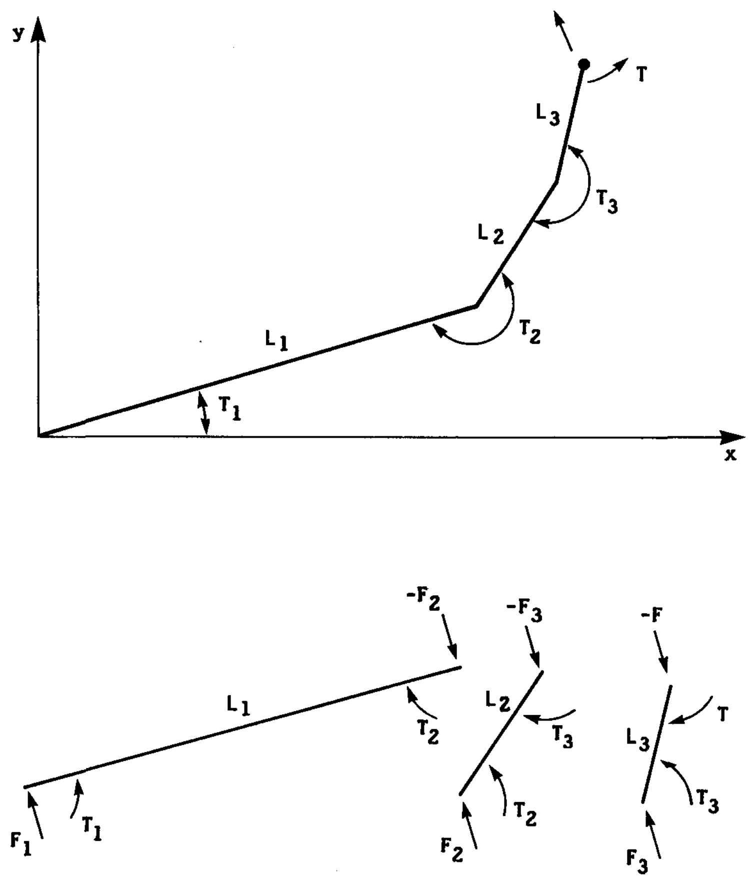
$$s_{23} = \sin(\theta_2 + \theta_3).$$

$$\underline{r}_1 = l_1[c_1, s_1]$$
$$\underline{r}_2 = l_2[c_{12}, s_{12}]$$
$$\underline{r}_3 = l_3[c_{123}, s_{123}]$$

and so

$$(\underline{r}_3) \times \underline{F} = (l_3c_{123})v - (l_3s_{123})u$$
$$(\underline{r}_2 + \underline{r}_3) \times \underline{F} = (l_2c_{12} + l_3c_{123})v - (l_2s_{12} + l_3s_{123})u$$
$$(\underline{r}_1 + \underline{r}_2 + \underline{r}_3) \times \underline{F}$$
$$= (l_1c_1 + l_2c_{12} + l_3c_{123})v - (l_1s_1 + l_2s_{12} + l_3s_{123})u$$

This can be written in matrix form.



y
x
T
L_3
T_3
L_2
T_2
L_1
T_1
$-F_2$
$-F_3$
$-F$
L_1
T_2
L_2
T_3
L_3
T
F_1
T_1
F_2
T_2
F_3
T_3

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12} + l_3 s_{123}) & (l_1 c_1 + l_2 c_{12} + l_3 c_{123}) & 1 \\ -(l_2 s_{12} + l_3 s_{123}) & (l_2 c_{12} + l_3 c_{123}) & 1 \\ -(l_3 s_{123}) & (l_3 c_{123}) & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ T \end{bmatrix}$$

This tells us how to calculate what motor torques are needed to apply a given force and torque to the object held by the manipulator. Notice that we could have arrived at this result by first considering the tip pinned in place only (that is, T=0) and then separately reason out that to apply torque T, each joint torque would have to be increased by T.

The determinant of the above matrix is $l_1 l_2 \sin(\theta_2)$. If θ_2 is neither 0 nor π, we can invert the matrix and solve for u, v, and T given the three joint torques.

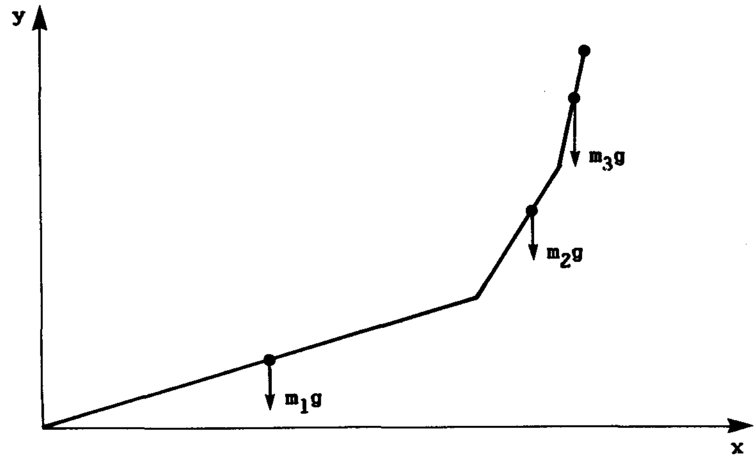
$$\begin{bmatrix} u \\ v \\ T \end{bmatrix} = \frac{1}{l_1 l_2 s_2} \begin{bmatrix} l_2 c_{12} & -(l_1 c_1 + l_2 c_{12}) & l_1 c_1 \\ l_2 s_{12} & -(l_1 s_1 + l_2 s_{12}) & l_1 s_1 \\ l_2 l_3 s_3 & -(l_1 l_3 s_{23} + l_2 l_3 s_3) & l_1 l_2 s_2 + l_1 l_3 s_{23} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$$

Gravity

Gravity is again very simple to take into account. If we assume that the center of mass of each link is in its geometric center we find that:

$$\underline{F}_3 = m_3 \underline{g}$$
$$\underline{F}_2 = (m_2 + m_3)\underline{g}$$
$$\underline{F}_1 = (m_1 + m_2 + m_3)\underline{g}$$

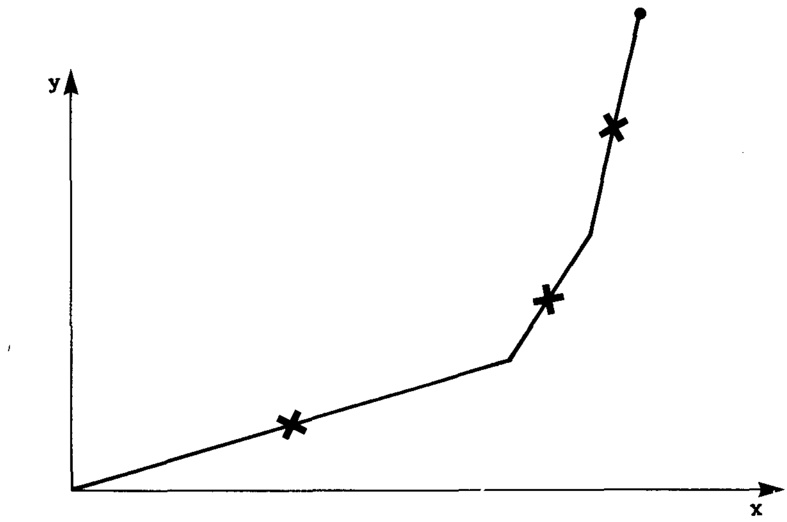
From these, we can derive the torques induced by gravity:



$$
\begin{aligned}
T_{3g} &= -m_3\ (1/2)\underline{r}_3 \times \underline{g} = g[(1/2)m_3 l_3 c_{123}] \\
T_{2g} &= T_{3g} - m_3\ (1/2)\underline{r}_2 \times \underline{g} - m_3\underline{r}_2 \times \underline{g} \\
&= g[(1/2)m_2 + m_3)l_2 c_{12} + (1/2)m_3 l_3 c_{123}] \\
T_{1g} &= T_{2g} - m_1\ (1/2)\underline{r}_1 \times \underline{g} - (m_2 + m_3)\underline{r}_1 \times \underline{g} \\
&= g[((1/2)m_1 + m_2 + m_3)l_1 c_1 \\
&\quad + ((1/2)m_2 + m_3)l_2 c_{12} + (1/2)m_3 l_3 c_{123}]
\end{aligned}
$$

Dynamics with Three Equal Links

For definiteness we will again consider a simple case where l_1, l_2, and l_3 are all equal to a length l. The more general case involves a lot more arithmetic and the form of the final result is the same, only numerical constants will be changed. Further, we will ignore gravity for now, and assume the links to be uniform sticks of mass m and inertia $(1/12)ml^2$ about their center of mass. Once again we start by finding the rotational and



translational velocities of each of the links. Evidently the angular velocities of the three links are $\dot{\theta}_1$, $(\dot{\theta}_1 + \dot{\theta}_2)$, and $(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)$.

The square of the magnitude of the instantaneous linear velocity of the center of mass of link 1 is simply

$$|(1/2)\underline{r}_1\dot{\theta}_1|^2 = l^2((1/4)\dot{\theta}_1{}^2)$$

For the square of the magnitude of the velocity of the center of link 2 we have

$$\begin{aligned}
&|\underline{r}_1\dot{\theta}_1 + (1/2)\underline{r}_2(\dot{\theta}_1 + \dot{\theta}_2)|^2 \\
&\quad = l^2[\dot{\theta}_1{}^2 + \cos(\theta_2)\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) + 1/4\ (\dot{\theta}_1 + \dot{\theta}_2)^2] \\
&\quad = l^2[\dot{\theta}_1{}^2(5/4 + \cos(\theta_2)) + \dot{\theta}_1\dot{\theta}_2((1/2) + \cos(\theta_2)) + \dot{\theta}_2{}^2\ (1/4)]
\end{aligned}$$

For the square of the magnitude of the velocity of the center of link 3 we have

$$
\begin{aligned}
&|\underline{r}_1\dot\theta_1 + \underline{r}_2(\dot\theta_1 + \dot\theta_2) + (1/2)\underline{r}_3(\dot\theta_1 + \dot\theta_2 + \dot\theta_3)|^2 \\
&\quad = 1^2[\dot\theta_1{}^2 + 2\cos(\theta_2)\dot\theta_1(\dot\theta_1 + \dot\theta_2) \\
&\qquad + (\dot\theta_1 + \dot\theta_2)^2 \\
&\qquad + \cos(\theta_3)(\dot\theta_1 + \dot\theta_2)(\dot\theta_1 + \dot\theta_2 + \dot\theta_3) \\
&\qquad + 1/4(\dot\theta_1 + \dot\theta_2 + \dot\theta_3)^2 \\
&\qquad + \cos(\theta_2 + \theta_3)\dot\theta_1(\dot\theta_1 + \dot\theta_2 + \dot\theta_3)] \\
&\quad = 1^2[\dot\theta_1{}^2\ (9/4 + 2\cos(\theta_2) + \cos(\theta_3) + \cos(\theta_2 + \theta_3)) \\
&\qquad + \dot\theta_1\dot\theta_2(11/2 + 2\cos(\theta_2) \\
&\qquad\qquad + 2\cos(\theta_3) + \cos(\theta_2 + \theta_3)) \\
&\qquad + \dot\theta_2{}^2(5/4 + \cos(\theta_3)) \\
&\qquad + \dot\theta_2\dot\theta_3(1/2 + \cos(\theta_3)) \\
&\qquad + \dot\theta_3{}^2(1/4) \\
&\qquad + \dot\theta_3\dot\theta_1(1/2 + \cos(\theta_3) + \cos(\theta_2 + \theta_3))]
\end{aligned}
$$

Isn't that lovely. We are now ready to add up the kinetic energy due to rotation and that due to linear translation of the center of mass for all three links to obtain the Lagrangian.

$$
\begin{aligned}
L = 1/2\ ml^2\ [&\dot\theta_1{}^2(4 + 3\cos(\theta_2) + \cos(\theta_2 + \theta_3) + \cos(\theta_3)) \\
&+ \dot\theta_1\dot\theta_2(19/3 + 3\cos(\theta_2) + \cos(\theta_2 + \theta_3) + 2\cos(\theta_3)) \\
&+ \dot\theta_2{}^2(5/3 + \cos(\theta_3)) \\
&+ \dot\theta_2\dot\theta_3(2/3 + \cos(\theta_3)) \\
&+ \dot\theta_3{}^2\ (1/3) \\
&+ \dot\theta_3\dot\theta_1(2/3 + \cos(\theta_2 + \theta_3) + \cos(\theta_3))]
\end{aligned}
$$

So this is the Lagrangian for this system and from it we will be able to calculate the relation between joint torques and joint accelerations. Let us use the shorthand notation for trigonometric terms introduced in the discussion of the statics of this device, e.g., $s_{23} = \sin(\theta_2 + \theta_3)$.

We will next derive the partial derivates of the Lagrangian with respect to θ_1, θ_2, θ_3, $\dot\theta_1$, $\dot\theta_2$, $\dot\theta_3$. Let

$L = 1/2\ ml^2 L'$ as before.

$$-\partial L'/\partial\theta_1 = 0$$
$$-\partial L'/\partial\theta_2 = \dot\theta_1^2(3\ s_2 + s_{23}) + \dot\theta_1\dot\theta_2(3\ s_2 + s_{23}) + \dot\theta_3\dot\theta_1(s_{23})$$
$$-\partial L'/\partial\theta_3 = \dot\theta_1^2(s_{23} + s_3) + \dot\theta_1\dot\theta_2(s_{23} + 2\ s_3) + \dot\theta_2^2(s_3) + \dot\theta_2\dot\theta_3(s_3) + \dot\theta_3\dot\theta_1(s_{23} + s_3)$$

The partial derivates of the Lagrangian with respect to angular velocity are:

$$\partial L'/\partial\dot\theta_1 = 2\dot\theta_1(4 + 3c_2 + c_{23} + c_3) + \dot\theta_2(19/3 + 3c_2 + c_{23} + 2c_3) + \dot\theta_3(2/3 + c_{23} + c_3)$$
$$\partial L'/\partial\dot\theta_2 = \dot\theta_1(19/3 + 3c_2 + c_{23} + 2c_3) + 2\dot\theta_2(5/3 + c_3) + \dot\theta_3(2/3 + c_3)$$
$$\partial L'/\partial\dot\theta_3 = \dot\theta_1(2/3 + c_{23} + c_3) + \dot\theta_2(2/3 + c_3) + 2\dot\theta_3(1/3)$$

Next we will need the time rate-of-change of the last three quantities above:

$$\begin{aligned}d/dt\ (\partial L'/\partial\dot\theta_1) = {} & 2\ddot\theta_1(4 + 3c_2 + c_{23} + c_3) \\ & + \ddot\theta_2(19/3 + 3c_2 + c_{23} + 2c_3) + \ddot\theta(2/3 + c_{23} + c_3) \\ & - 2\dot\theta_1(3s_2\dot\theta_2 - s_{23}(\dot\theta_2 + \dot\theta_3) - s_3\dot\theta_3) \\ & - \dot\theta_2(3s_2\dot\theta_2 + s_{23}(\dot\theta_2 + \dot\theta_3) - 2s_3\dot\theta_3) \\ & - \dot\theta_3(s_{23}(\dot\theta_2 + \dot\theta_3) - s_3\ \dot\theta_3)\end{aligned}$$
$$\begin{aligned}d/dt(\partial L'/\partial\dot\theta_2) = {} & \ddot\theta_1(19/3 + 3c_2 + c_{23} + 2c_3) \\ & + 2\ddot\theta_2(5/3 + c_3) + \ddot\theta_3(2/3 + c_3) \\ & - \dot\theta_1(3s_2\dot\theta_2 - s_{23}(\dot\theta_2 + \dot\theta_3) - s_3\dot\theta_3) - 2\dot\theta_2(s_3\dot\theta_3) - \dot\theta(s_3\dot\theta_3)\end{aligned}$$
$$\begin{aligned}d/dt\ (\partial L'/\partial\dot\theta_3) = {} & \ddot\theta_1(2/3 + c_{23} + c_3) + \ddot\theta_2(2/3 + c_3) + 2\ddot\theta_3(1/3) \\ & - \dot\theta_1(s_{23}(\dot\theta_2 + \dot\theta_3) - s_3\dot\theta_3) - \dot\theta_2(s_3\dot\theta_3)\end{aligned}$$

Finally, inserting these terms into Lagrange's equation gives:

$$
\begin{aligned}
T_1' &= d/dt(L'/\partial\dot\theta_1) - \partial L'/\partial\theta_1 \\
&= 2\ddot\theta_1\,(4 + 3c_2 + c_{23} + c_3) \\
&\quad + \ddot\theta_2(19/3 + 3c_2 + c_{23} + 2c_3) \\
&\quad + \ddot\theta_3(2/3 + c_{23} + c_3) \\
&\quad - \dot\theta_1\dot\theta_2(6s_2 + 2s_{23}) - \dot\theta_2{}^2(3s_2 + s_{23}) \\
&\quad - \dot\theta_2\dot\theta_3(2s_{23}) \\
&\quad - \dot\theta_3{}^2(s_{23} + s_3) - \dot\theta_3\dot\theta_1(2s_{23} + s_3) \\
T_2' &= d/dt(\partial L'/\partial\dot\theta_2) - \partial L'/\partial\theta_2 \\
&= \ddot\theta_1(19/3 + 3c_2 + c_{23} + 2c_3) + 2\ddot\theta_2(5/3 + c_3) + \ddot\theta_3(2/3 + c_3) \\
&\quad + \dot\theta_1{}^2(3s_2 + s_{23}) - \dot\theta_2\dot\theta_3(2s_3) - \dot\theta_3{}^2(s_3) - \dot\theta_3\theta_1(2s_3) \\
T_3' &= d/dt(\partial L'/\partial\dot\theta_3) - \partial L'/\partial\theta_3 \\
&= \ddot\theta_1(2/3 + c_{23} + c_3) + \ddot\theta_2(2/3 + c_3) + \ddot\theta_3(2/3) \\
&\quad + \dot\theta_1{}^2(s_{23} + s_3) + \dot\theta_1\dot\theta_2(2\,s_3) + \dot\theta_2{}^2(s_3)
\end{aligned}
$$

As in the two-link case, the equations above can be expressed in matrix terms. The torque vector is equal to a sensitivity matrix times the angular acceleration vector, plus a vector of torques due to velocity products. The sensitivity matrix is symmetric, and its diagonal elements are always positive. The terms in this matrix depend only on the joint angles because all the velocity-product terms are segregated out.

Given the arm state (joint angles and joint angular velocities), we can calculate what torques need to be applied to each of the joints in order to achieve a given angular acceleration for each of the joints. We only need to invert the sensitivity matrix.

Extensions to Three Dimensions

Once the basic principles are understood, we can proceed to introduce the extensions necessary to deal with manipulators in three dimensions. There is little difficulty as regards position and force since in an n-dimensional space these quantities can be conveniently represented by n-dimensional vectors. A general position or force generator will need n degrees of freedom.

Unfortunately we are not so lucky with orientation and torque. These can not be usefully thought of as vectors. For example, in three dimensions we know that rotations don't commute, while vector addition does. It is a misleading coincidence that it takes three variables to specify a rotation in three dimensions.

It takes $n(n-1)/2$ variables to specify a rotation in n-dimensional space. Why? A general rotation can be made up of components each of which carries one axis part way towards a second axis. There are n axes, and so "n choose 2" distinct pairs of axes. There are therefore that number of "elementary" rotations. It is not correct to think of rotations "about an axis"; in our two-dimensional example such rotations would carry one out of the plane of the paper, and in four dimensions, not all possible rotations would be generated by considering only combinations of the four rotations about the coordinate axes.

Another way of approaching this problem is to look at matrices that represent coordinate transformations that correspond to rotations. Such matrices are ortho-normal and of size $n*n$. How many of the n^2 entries can be freely chosen? The condition of normality generates n constraints, and the condition of orthogonality another $n(n-1)/2$. So we have

$$n^2 - n - n(n-1)/2 = n(n-1)/2$$

degrees of freedom left.

To specify position and orientation, or force and torque in n dimensions requires $n(n-1)/2+n$ variables. A general purpose n-dimensional manipulator thus needs to have $n(n+1)/2$ degrees of freedom. For $n=3$, this is 6. The coincidence that it takes 3 variables to specify a rotation in three dimensions allows some simplifications. A torque, for example, can be calculated by taking cross-products. In higher dimensions, one needs to look at exterior tensor products. A useful way of specifying rotations in three dimensions is by means of Euler angles: roll, pitch, and yaw, for example. It is straightforward to convert between this representation and the ortho-normal matrix notation.

Kinematics

It is no longer sufficient to represent each link as a vector, since the joints at its two ends may have axes that are not parallel. The way to deal with this problem is simply to erect a coordinate system fixed to each link. Corresponding to each joint there will be a coordinate transformation from one system to the next. This transformation can be represented by a 3x3 rotation matrix plus an offset vector. It is convenient to combine these into one 4x4 transformation matrix that has (0 0 0 1) as its last row. This allows one easily to invert the transformation, so as to allow conversion of coordinates in the other direction as well.

The entries in this matrix will be trigonometric polynomials in the joint angles. In order to determine the relation between links separated by more than one joint, one can simply multiply the transformation matrices corresponding to the intervening joints. Doing this for the complete manipulator, one obtains a single matrix that allows one to relate coordinates relative to the tip or terminal device to coordinates relative to the base of the device. In fact the 3x3 rotation submatrix gives us the rotation of the last link relative to the base and and hence its orientation, while the offset 3x1 submatrix is the position of the tip of the last link with respect to the base.

Given the joint variables, it is then a relatively straightforward matter to arrive at the position and orientation of the terminal device or tip. These values are of course unique for a particular set of joint variables.

The Inverse Problem in Three Dimensions is Intractable

Unfortunately the inversion is much harder. One way to approach this problem would be to consider the 3x3 rotation submatrix made up entirely of polynomials in sines and cosines of joint angles and the 3x1 offset submatrix which contains link-lengths as well and try to solve for the sines and cosines of the six joint angles. There are twelve equations in twelve unknowns, so we expect there to be a finite number of solutions. When solving polynomial equations by eliminating variables the degree of the resulting polynomials grows as the product of the polynomials combined. We could easily end up with one polynomial in one unknown with a degree of several thousand. So in general this problem is intractable.

There are a number of conditions on the link geometry that make this problem solvable by noniterative techniques. Several such configurations are known, but one of the easiest to explain involves decoupling the orientation from the position. One then has to solve two problems that are much smaller, each having only three degrees of freedom. Suppose for example that the last three rotational joints intersect in one point, call it the wrist. Then these last three can take care of the orientation, while the remaining three position the wrist. Given the orientation of the last link it is easy to calculate where the wrist should be relative to the tip position. Given the position of the wrist one can solve the inversion problem for the first three links. This can usually be done by careful inspection rather than blind solution of trigonometric polynomials. Often also the first three links are simply a combination of the two-link geometry we have already solved and an offset polar-coordinate problem.

Now that we know the first three joint angles we can calculate the orientation of the third to which the wrist attaches. Comparing this with the last link, it is simple to calculate the three wrist angles by matrix multiplication and solving for the Euler angles appropriate to the design of the wrist.

Statics

By controlling the six joint torques we can produce a given force and torque at the terminal device. The same coordinate transformation matrices used for solving the kinematics prove useful here. Cross-products give us the required torques, with joint motors supporting the components around the joint axes, while the pin joints transmit the other components. The calculations are straightforward.

Gravity compensation calculations also follow the familiar pattern. In many cases manipulators intended for positional control have been used to generate forces and torques in a different manner. The idea is to use the inherent compliance of the device as a kind of spring and to drive the joints to angles slightly away from the equilibrium position. Since the stress-strain matrix of such a device is very complex and it has different spring constants in different directions, as well as coupling between forces and torques, this technique on its own is not very useful. One solution relies on a force and torque sensor in the wrist. From the output of such a device one can calculate the forces and torques at the tip and servo the joint angles accordingly. The advantage of this technique is that friction in the first three joints does not corrupt the result and that the measurement is made beyond the point where the heaviest and stickiest components of the manipulator are.

Dynamics

The main additional difficulty of manipulators in higher dimensions is that inertia too now has several components instead of just one. The dynamic behavior of a rigid body as regards rotation can be conveniently expressed as a symmetrical, square inertia matrix. This relates the applied torque components to the resulting angular accelerations. The same general idea carries through, with the distinction that the calculations get very messy and have to be approached in a systematic fashion. A practical difficulty is the measurement of the components of the inertia matrices for each of the links of the manipulator.

Of course the problem is somewhat complicated should the manipulator actually manipulate something.

References

B. K. P. Horn and H. Inoue, *Kinematics of the MIT-AI-Vicarm Manipulator*, MIT AI Laboratory Working Paper 69, 1974.

M. E. Kahn, *The Near-minimum-time Control of Open-loop Articulated Kinematic Chains*, Stanford Artificial Intelligence Memo 106, 1969.

R. Paul, *Modelling, Trajectory Calculation and Servoing of a Computer Controlled Arm*, Stanford Artificial Intelligence Memo 177, 1972.

D. L. Pieper, *The Kinematics of Manipulators Under Computer Control*, Stanford Artificial Intelligence Memo 72, 1968.

J. J. Uicker, *On the Dynamic Analysis of Spatial Linkages using 4*4 Matrices*, PhD Dissertation, Northwestern University, 1965.

J. J. Uicker,"Dynamic Force Analysis of Spatial Linkages", *Transactions ASME*, 1967.

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