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THE CURVE OF LEAST ENERGY

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Abstract: Here we search for the curve which has the smallest integral of the square of curvature, while passing through two given points with given orientation. This is the true shape of a spline used in lofting. In computer-aided design, curves have been sought which maximize "smoothness". The curve discussed here is the one arising in this way from a commonly used measure of smoothness. The human visual system may use such a curve when it constructs a subjective contour.

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Introduction

The curve which passes through two specified points with specified orientation while minimizing

$$\mathcal{E} = \int \kappa^2 \, ds,$$

where κ is the curvature and s the arc-distance, has a number of interesting applications.

In a thin beam, curvature at a point is proportional to the bending moment [pg. 80, 1]. The total elastic energy stored in a thin beam is therefore proportional to the integral of the square of the curvature [pg. 163, 1]. The shape taken on by a thin beam is the one which minimizes its internal strain energy. This is why we call the curve sought here the minimum energy curve. A thin metal or wooden strip used by a drafts(wo)man to smoothly connect a number of points is called a spline [pg. 156, 2]. Such splines are used in creating lofted surfaces from plane parallel cross-sections of ship hulls and aircraft fuselages [pg. 228, 2]. The shape of a spline constrained to pass through two specified points with specified orientation is what we are after here.

In computer graphics and computer aided design there is a search for curves which are particularly "smooth" [pg. 156, 2; pg. 49, 3; pg. 119, 4; pg. 309, 5; pg. 43, 6]. One measure of smoothness is the inverse of the integral given above. Typically, cubic polynomial approximations are used instead of the optimal curve [pg. 162, 2; pg. 66, 3; pg. 129, 4; pg. 315, 5]. (Unfortunately, these approximations are called splines too).

It has been suggested that the human visual system uses a curve of low energy when completing a contour. Ullman [pg. 1, 7] proposes that a subjective contour consists of two circular arcs tangent at their point of contact. Out of the one-parameter family of solutions of this form he picks the one which minimizes the integral of the square of curvature [pg. 2, 7]. He notes that the curve so constructed may have near-minimal energy. (This does not mean that it necessarily lies close to the curve of minimum energy, as we shall see.) Brady *et al* used cubic polynomial approximations instead [8].

Preview

We will first consider a special case. Here the curve must pass through the points $(-1,0)$ and $(+1,0)$ in the xy-plane with vertical orientation at both points. We first determine the optimum curve as the limit of a series of approximations. This approach helps suggest algorithms for computing approximations to the ideal curve. Later we solve the variational problem directly. We then discuss how this curve can be translated, rotated, and scaled to produce a four parameter family of curves which contains the general solution.

A Semicircle

One curve which connects the two points $(-1,0)$ and $(+1,0)$, and has the desired orientation at these points, is a semicircle of radius one with the center at the origin (see Figure 1). The curvature equals one at all points and so the relevant integral for the section in the right hand quadrant becomes

$$\mathcal{E} = \int_0^{\pi/2} 1 \, ds = \pi/2.$$

The arc length happens to have the same value,

$$\mathcal{S} = \int_0^{\pi/2} ds = \pi/2,$$

while the maximum height of the curve above the x-axis is

$$\mathcal{H} = 1.$$

Can we do better, that is, find a curve with a smaller value of $\mathcal{E}$?

Two Circular Arcs

We try a combination of two circular arcs for the portion of the curve in the first quadrant, as shown in Figure 2. (The other half of the curve is obtained by reflection about the y-axis.) Let the first arc have radius R and angular extend θ, while the remaining portion has radius r and angular extend $(\pi/2\text{-}\theta)$. Note that the parameters r, R, and θ are not independent, since one can obtain from the diagram

$$(R-r)\cos\theta = (R-1).$$

The arc length of the right-hand portion of the curve becomes

$$\mathcal{S} = \theta R + (\pi/2 - \theta)\, r,$$

while the energy is

$$\mathcal{E} = \frac{\theta}{R} + \frac{(\pi/2 - \theta)}{r}.$$

Minimizing $\mathcal{E}$, subject to the given constraint, using the method of Lagrangian multipliers, leads to the set of equations

$$\frac{\theta}{R^2} + \frac{(\pi/2 - \theta)}{r^2}(1 - \sec\theta) = 0,$$

$$\frac{1}{R} - \frac{1}{r} + \frac{(\pi/2 - \theta)(R-r)}{r^2}\tan\theta = 0,$$

$$(R-r)\cos\theta - (R-1) = 0.$$

The second of these can be simplified into

$$r = R\,(\pi/2 - \theta)\tan\theta,$$

which, when applied to the first equation, yields

$$\theta\,(\pi/2 - \theta)(1 + \sec\theta) = 1.$$

Solving this numerically we obtain $\theta = 0.412868765...$

We can also show that $R = \dfrac{1}{[(1-\cos\theta) + (\pi/2 - \theta)\sin\theta]}$,

so that $R = 1.8227161...$

and $r = 0.92452847...$

Finally, since here $$\mathcal{H} = r + (R - r)\sin\theta$$

we see that $$\mathcal{H} = 1.2849161...$$

and $$\mathcal{E} = 1.4789649... \approx 0.94153834*(\pi/2),$$

while $$\mathcal{S} = 1.8230795... \approx 1.1606084*(\pi/2).$$

The energy in this curve is only 94.15% of that in the semicircle, so we *can* do better.

The Best Ellipse

The two-arc solution suggests that the optimum curve is elongated and has radius of curvature smaller than one at its peak, and larger than one near the x-axis. Is it an ellipse? The equation of the ellipse [pg. 411, 13; pg. 72, 14] shown in Figure 3 is

$$(x/a)^2 + (y/b)^2 = 1,$$

or in parametric form, $$x = a\cos t, \quad y = b\sin t.$$

The eccentricity e is defined by the equation

$$e^2 = 1 - (a/b)^2.$$

The arc length can be found as follows [pg. 26, 9],

$$\mathcal{S} = \int ds = \int_0^{\pi/2} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

where $$\dot{x} = \frac{dx}{dt} \text{ and } \dot{y} = \frac{dy}{dt}.$$

Now, $$\dot{x}^2 + \dot{y}^2 = a^2\sin^2 t + b^2\cos^2 t,$$

or, $$\dot{x}^2 + \dot{y}^2 = b^2[1 - e^2\sin^2 t].$$

So $$\mathcal{S} = b\int_0^{\pi/2} \sqrt{1 - e^2\sin^2 t}\, dt = b\,E(e),$$

where $E(e)$ is the complete elliptic integral of the second kind[1] [pg. 16, 9; pg. 833, 10; pg. 904, 11; pg. 589, 12]. (The elliptic integrals got their name from the fact that they first appeared in the mensuration of the ellipse.) Finally, for $a=1$,

$$\mathcal{S} = E(e) / \sqrt{1 - e^2}.$$

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1. Note that E denotes two things in this paper. When it has one argument,as here, it signifies the *complete* elliptic integral of the second kind.

The curvature can be found as follows [pg. 553, 13; pg. 22, 14]:

$$\kappa = \frac{\ddot{y}\,\dot{x} - \dot{y}\,\ddot{x}}{[\dot{x}^2 + \dot{y}^2]^{3/2}} = \frac{ab}{[a^2 \sin^2 t + b^2 \cos^2 t]^{3/2}}.$$

Now
$$\mathcal{E} = \int \kappa^2 \, ds = \int_0^{\pi/2} \kappa^2 \, [\dot{x}^2 + \dot{y}^2]^{1/2} dt,$$

So
$$\mathcal{E} = \frac{a^2}{b^3} \int_0^{\pi/2} \frac{1}{[1 - e^2 \sin^2 t]^{5/2}} \, dt.$$

The definite integral can be shown [using pg. 165, 11] to be equal to

$$[2\,(e'^2 + 1)\,E(e) - e'^2 K(e)] \,/\, (3\,e'^4),$$

where
$$e^2 + e'^2 = 1,$$

and $K(e)$ is the complete elliptic integral of the first kind [pg. 16, 9; pg. 834, 10; pg. 904, 11; pg. 589, 12]. So finally when $a = 1$,

$$\mathcal{E} = [2\,(2 - e^2)\,E(e) - (1 - e^2)\,K(e)] \,/\, [3 \sqrt{1 - e^2}\,].$$

When we set $e = 0$ we obtain $\mathcal{E} = \pi/2$, as we should, since the curve in this case is just a semicircle.

In order to find the best possible ellipse we need to differentiate the expression above with respect to e. Here we need the following derivatives [pg. 21, 9; pg. 907, 11]:

$$\frac{dE(k)}{dk} = \frac{E(k) - K(k)}{k},$$

$$\frac{dK(k)}{dk} = \frac{E(k) - k'^2\,K(k)}{k\,k'^2},$$

where
$$k^2 + k'^2 = 1.$$

The eccentricity of the optimum ellipse satisfies

$$E(e)\,[4e^4 - 5e^2 + 3] = K(e)\,[2e^4 - 5e^2 + 3].$$

Solving this equation numerically leads to

$$e = 0.6530018...$$

with
$$\mathcal{H} = b/a = 1 \,/\, \sqrt{1 - e^2} = 1.3203823...$$

$$\mathcal{E} = 1.4674751... \approx 0.93422368 * (\pi/2),$$

$$\mathcal{S} = 1.8311202\ldots \approx 1.1657273 * (\pi/2).$$

The maximum and minimum radii of curvature are

$$r_{max} = b^2/a = 1.7434096\ldots \text{ and } r_{min} = a^2/b = 0.75735636\ldots$$

This curve has an energy which is only 93.42% of that of the semicircle. We have found a curve which has a smaller value of $\mathcal{E}$ than our two-arc solution.

Can we do better still?

Multi-arc Approximation

Consider a smooth curve constructed out of n circular arcs (see Figure 4). Let the radius of curvature of the piece turning through the angle from α_i to α_{i+1} be r_i. We note that the total arc length, $\mathcal{S}'$, and the integral of the square of the curvature, $\mathcal{E}'$, are given by

$$\mathcal{S}' = \sum_{i=0}^{n-1} (\alpha_{i+1} - \alpha_i)\, r_i = r_{n-1}\alpha_n + \sum_{i=1}^{n-1} (r_{i-1} - r_i)\, \alpha_i,$$

$$\mathcal{E}' = \sum_{i=0}^{n-1} \frac{(\alpha_{i+1} - \alpha_i)}{r_i} = \frac{\alpha_n}{r_{n-1}} - \sum_{i=1}^{n-1} \left[\frac{1}{r_i} - \frac{1}{r_{i-1}}\right]\alpha_i,$$

where $\alpha_0 = 0$ and $\alpha_n = \pi/2$. We also have to compute the width, $\mathcal{W}'$, and height, $\mathcal{H}'$,

$$\mathcal{W}' = \sum_{i=0}^{n-1} r_i (\cos\alpha_i - \cos\alpha_{i+1}) = r_0 \cos\alpha_0 - \sum_{i=1}^{n-1} (r_{i-1} - r_i) \cos\alpha_i,$$

$$\mathcal{H}' = \sum_{i=0}^{n-1} r_i (\sin\alpha_{i+1} - \sin\alpha_i) = r_{n-1} \sin\alpha_n + \sum_{i=1}^{n-1} (r_{i-1} - r_i) \sin\alpha_i.$$

To solve our original problem we need to scale whatever curve we obtain so that its overall width equals 2 instead of $2\mathcal{W}'$. The scaled values are as follows:

$$\mathcal{H} = \mathcal{H}'/\mathcal{W}',$$

$$\mathcal{E} = \mathcal{E}'\,\mathcal{W}',$$

$$\mathcal{S} = \mathcal{S}'/\mathcal{W}'.$$

Note that the integral of the square of the curvature is *decreased* when we make the curve larger without changing its shape.

Optimum Multi-arc Approximation

Our task now is clear: We have to minimize $\mathcal{E}'\mathcal{W}'$ by suitable choices of the parameters r_i (for $i=0,1 \ldots n-1$) and α_i (for $i=1,2 \ldots n-1$). Actually, we can pick one of the radii arbitrarily, r_0 for example, since the whole curve will simply be scaled accordingly. The minimization looks difficult at first when one considers the complexity of the product $\mathcal{E}'\mathcal{W}'$ and its derivatives. It appears necessary to resort to numerical techniques to solve for the $2(n-1)$ parameters.

Fortunately this is not the case, for if

$$\frac{\partial}{\partial \chi}(\mathcal{E}'\mathcal{W}') = 0,$$

then, by the rule for the differentiation of a product,

$$-\frac{\partial \mathcal{W}'}{\partial \chi} / \frac{\partial \mathcal{E}'}{\partial \chi} = \mathcal{W}'/\mathcal{E}',$$

for arbitrary χ (i. e. r_i and α_i). Since the right-hand side is independent of χ, it must equal a (positive) constant, c^2 say. Thus we find that

$$-\frac{\partial \mathcal{W}'}{\partial \alpha_i} / \frac{\partial \mathcal{E}'}{\partial \alpha_i} = c^2 \text{ for } i = 1,2 \ldots n-1,$$

$$-\frac{\partial \mathcal{W}'}{\partial r_i} / \frac{\partial \mathcal{E}'}{\partial r_i} = c^2 \text{ for } i = 0,1 \ldots n-1.$$

We need the following derivatives now,

$$\frac{\partial \mathcal{E}'}{\partial \alpha_i} = -\left[\frac{1}{r_i} - \frac{1}{r_{i-1}}\right] \text{ for } i = 1,2 \ldots n-1,$$

$$\frac{\partial \mathcal{E}'}{\partial r_i} = -\frac{(\alpha_{i+1} - \alpha_i)}{r_i^2} \text{ for } i = 0,1 \ldots n-1,$$

$$\frac{\partial \mathcal{W}'}{\partial \alpha_i} = (r_{i-1} - r_i) \sin \alpha_i \text{ for } i = 1,2 \ldots n-1,$$

$$\frac{\partial \mathcal{W}'}{\partial r_i} = (\cos \alpha_i - \cos \alpha_{i+1}) \text{ for } i = 0,1 \ldots n-1.$$

Using these derivatives in the equations above we obtain

$$r_i\, r_{i-1} \sin \alpha_i = c^2 \text{ for } i = 1,2 \ldots n-1,$$

$$-r_i^2 \frac{\cos \alpha_{i+1} - \cos \alpha_i}{\alpha_{i+1} - \alpha_i} = c^2 \text{ for } i = 0,1 \ldots n-1.$$

It is easy to verify that in the case that $n=2$, we obtain the same equations as before, provided we introduce the additional constraint $\mathcal{W}'=1$ or

$$r_0 - (r_0 - r_1)\cos\alpha_1 = 1.$$

Note that, if c is known, a simple procedure will give us all of the parameters. Let $r_0 = 1$, say, then the second equation can be used to find α_1. (This non-linear equation has to be solved numerically.) The first equation then allows one to solve for r_1. Knowing r_1, the second equation allows one to find α_2, and so on. If the value of c is correct, the process will terminate with $\alpha_n = \pi/2$. The correct solution can be found by searching for the appropriate value of c. This is very much simpler than a direct search on the $2(n-1)$ parameters.

Some Helpful Relationships

A number of interesting observations can be made now about the multi-arc solution. First of all, the "energy" $\delta\mathcal{E}'$ in an individual arc is directly proportional to the projection $\delta\mathcal{W}'$ of this arc on the x-axis, since

$$\delta\mathcal{E}' = \frac{(\alpha_{i+1} - \alpha_i)}{r_i},$$

and

$$\delta\mathcal{W}' = r_i(\cos\alpha_i - \cos\alpha_{i+1}).$$

So we have

$$\delta\mathcal{E}'/\delta\mathcal{W}' = 1/c^2,$$

and we already know, of course, that

$$\mathcal{E}'/\mathcal{W}' = 1/c^2.$$

Next, notice that the projection, $\delta\mathcal{H}'$, of the $(i+1)$-th arc on the y-axis equals

$$\delta\mathcal{H}' = r_i(\sin\alpha_{i+1} - \sin\alpha_i) = c^2\left[\frac{1}{r_{i+1}} - \frac{1}{r_{i-1}}\right].$$

The height at the tip of the i-th arc then is

$$\mathcal{H}_i' = c^2\left[\frac{1}{r_i} + \frac{1}{r_{i-1}} - \frac{1}{r_0}\right] \text{ for } i = 1,2 \ldots n-1,$$

since

$$\mathcal{H}_1' = r_0 \sin\alpha_1 = c^2/r_1.$$

Also

$$\mathcal{H}' = c^2\left[\frac{1}{r_{n-1}} - \frac{1}{r_0}\right] + r_{n-1}.$$

Three, Four and Five Arcs

The optimum solution for three arcs gives

$$\mathcal{H} = 1.399926...$$

$$\mathcal{E} = 1.456879... \approx 0.9274780*(\pi/2)$$

$$\mathcal{S} = 1.929128... \approx 1.228121*(\pi/2),$$

and shows us that the ellipse is not optimal after all. For four arcs we find

$$\mathcal{H} = 1.462089...$$

$$\mathcal{E} = 1.448212... \approx 0.9219604*(\pi/2)$$

$$\mathcal{S} = 1.987501... \approx 1.265282*(\pi/2),$$

and for five arcs we get

$$\mathcal{H} = 1.500993...$$

$$\mathcal{E} = 1.443930... \approx 0.9192345*(\pi/2)$$

$$\mathcal{S} = 2.024437... \approx 1.288796*(\pi/2).$$

These solutions are shown in Figure 5. We see that $\mathcal{E}$ is dropping more and more slowly, while $\mathcal{S}$ is growing, as is $\mathcal{H}$.

For five arcs, the parameters for the unscaled curve ($r_0 = 1$), are as follows

$$r_1 = 0.500258... \qquad \alpha_1 = 0.078707...$$
$$r_2 = 0.334497... \qquad \alpha_2 = 0.237276...$$
$$r_3 = 0.253163... \qquad \alpha_3 = 0.483045...$$
$$r_4 = 0.207337... \qquad \alpha_4 = 0.847067...$$
$$r_5 = 0.189705... \qquad \alpha_5 = 1.57079...$$

Perhaps we can guess the true minimum energy curve from the numerical data obtained so far. The parameters seem to roughly fit into a pattern like

$$r_i = \frac{1}{(i+1)} \text{ and } \alpha_i = \frac{\pi}{2}\frac{i(i+1)}{n(n+1)}.$$

In this case

$$(\alpha_{i+1} - \alpha_i) = \pi\frac{i+1}{n(n+1)},$$

so the arc lengths are

$$\delta\mathcal{S}' = r_i(\alpha_{i+1} - \alpha_i) = \frac{\pi}{n(n+1)}.$$

That is, the arcs all have the same length, and curvature increases linearly along the curve.

Now we could sum the series for arc length and energy (easy) and for height and width (hard). Then we would discover that $\mathcal{E} = \mathcal{E}'\,\mathcal{W}'$ decreases with n, and we could find its limit as the number of arcs tends to infinity. Instead, we proceed directly to the curve obtained in the limiting process.

The Cornu Spiral

The curve which has curvature varying linearly with arc-length is called the Cornu Spiral (or Euler's Spiral) [pg. 190, 2; pg. 190, 14]. It can be defined using the two Fresnel integrals [pg. 820, 10; pg. 930, 11; pg. 300, 12].

$$C(s) = \int_0^s \cos\left(\frac{\pi}{2}t^2\right) dt,$$

$$S(s) = \int_0^s \sin\left(\frac{\pi}{2}t^2\right) dt.$$

If we let

$$x = C(s) \text{ and } y = S(s),$$

we obtain a curve starting at the origin and curling upwards in the first quadrant. We note that

$$\dot{x} = \cos\left(\frac{\pi}{2}s^2\right) \text{ and } \dot{y} = \sin\left(\frac{\pi}{2}s^2\right),$$

$$\ddot{x} = -\pi s \sin\left(\frac{\pi}{2}s^2\right) \text{ and } \ddot{y} = \pi s \cos\left(\frac{\pi}{2}s^2\right).$$

This verifies that s is the arc-length along this curve, since

$$\dot{x}^2 + \dot{y}^2 = 1,$$

and that the curvature varies linearly with arc length, since

$$\ddot{y}\,\dot{x} - \ddot{x}\,\dot{y} = \pi\, s.$$

The part of the spiral of interest to us here extends to the right up to the point where the curve becomes vertical, that is, $\dot{x}=0$. This is the point where $s=1$ and

$$x = C(1) = 0.7798934\ldots \text{ and } y = S(1) = 0.4382591\ldots$$

The energy in this portion of the spiral is just

$$\mathcal{E}' = \int_0^1 \kappa^2 ds = \int_0^1 (\pi s)^2 ds = \pi^2/3.$$

We now build a smooth curve connecting the two points $(-1,0)$ and $(+1,0)$ by scaling, rotating, and shifting this tendril as shown in Figure 6. In the right hand quadrant we use

$$x = 1 - \frac{S(s)}{S(1)} \text{ and } y = \frac{C(s)}{S(1)}.$$

(The rest of the curve is obtained by reflection about the y-axis.)

We then find that

$$\mathcal{H} = C(1)/S(1) = 1.779525...$$

$$\mathcal{E} = (\pi^2/3)\, S(1) = 1.441814... \approx 0.9178877*(\pi/2),$$

$$\mathcal{F} = 1/S(1) = 2.281755... \approx 1.452610*(\pi/2).$$

The curve constructed out of a portion of the Cornu Spiral only has 91.78% of the energy of the semicircle and is thus the best curve so far.

But, can we do better still?

Six Arcs and More

Unfortunately, the Cornu Spiral is not optimal either, as one sees by considering the best six-arc solution for which

$$\mathcal{E} = 1.441508... \approx 0.9176931*(\pi/2).$$

For eight arcs, $\mathcal{E}_8 \approx 0.916097*(\pi/2)$,
for sixteen arcs, $\mathcal{E}_{16} \approx 0.914532*(\pi/2)$,
for thirty-two arcs, $\mathcal{E}_{32} \approx 0.914285*(\pi/2)$,
and for sixty-four $\mathcal{E}_{64} \approx 0.913953*(\pi/2)$.

These solutions are shown in Figure 7.

It seems that the total energy is approaching some limit, near 91.39% of that in the semicircle. Some of these results are summarized in Tables I and II.

TABLE I

n	$\mathcal{E}/(\pi/2)$	$\mathcal{F}/(\pi/2)$	$\mathcal{H}$	r_{min}
1	1.0	1.0	1.0	1.0
2	.9415383	1.160608	1.284916	.9245284
3	.9274780	1.228121	1.399925	.8950140
4	.9219604	1.265282	1.462089	.8794932
5	.9192345	1.288796	1.500993	.8700129
6	.9176896	1.305012	1.527622	.8636634
7	.9167300	1.316870	1.546987	.8591353
8	.9160932	1.325918	1.561701	.8557555

TABLE II

n	$\mathcal{E}/(\pi/2)$	$\mathcal{F}/(\pi/2)$	$\mathcal{H}$	r_{min}
16	.9144692	1.358673	1.614492	.8442353
32	.9140406	1.375704	1.641625	.8389106
64	.9139305	1.384403	1.655392	.836516
128	.9139025	1.388789	1.662308	.835461
256	.9138955	1.390998	1.665786	.834998
512	.9138938	1.392110	1.667534	.83473
1024	.9138934	1.392671	1.668417	.83467
2048	.9138932	1.392984	1.668908	.83463

We can get better and better approximations to the optimum curve, provided we also carry out computations with more and more significant figures. Note, by the way, that while $\mathcal{E}$ varies little once n is reasonably large, $\mathcal{F}$ and $\mathcal{H}$ continue to show appreciable changes. This is a reflection of the fact that some distortions of the optimum curve produce only small changes in the total energy.

Some Observations About the Optimum Curve

The multi-arc approximation tends to the optimum curve in the limit as n tends to infinity. So we can learn some properties of the optimum curve from what we have so far. First of all,

from $$r_i r_{i-1} \sin \alpha_i = c^2$$

we get $$c^2 \kappa^2 = \cos \psi,$$

where κ is the curvature and ψ is the angle which the curve makes with the x-axis. The constant c only affects the size of the curve, not its shape. We cannot determine it at this point.

From $$-r_i^2 \frac{\cos \alpha_{i+1} - \cos \alpha_i}{\alpha_{i+1} - \alpha_i} = c^2,$$

we get $$r_i^2 \frac{\sin [(\alpha_{i+1} - \alpha_i)/2]}{(\alpha_{i+1} - \alpha_i)/2} \sin [(\alpha_{i+1} + \alpha_i)/2] = c^2,$$

which in the limit again leads to $$c^2 \kappa^2 = \cos \psi.$$

We also obtain $$\frac{d\mathcal{E}}{dx} = 1/c^2,$$

from $$\frac{\delta \mathcal{E}'}{\delta \mathcal{W}'} = 1/c^2.$$

Now $$\mathcal{E} = \int \kappa^2 \, ds = \int \kappa^2 \sqrt{1 + (dy/dx)^2} \, dx,$$

so $$\kappa^2 \sqrt{1 + (dy/dx)^2} = 1/c^2.$$

Further, since, $$\frac{dy}{dx} = \tan \psi,$$

we again obtain, $$c^2\kappa^2 = \cos \psi.$$

Each of these approaches leads us to the same simple differential equation for the curve.

Finally, from $$\mathcal{H}_i{}' = c^2\left[\frac{1}{r_i} + \frac{1}{r_{i-1}} - \frac{1}{r_0}\right],$$

we get in the limit $$-\kappa = y/2c^2.$$

That is, the curvature varies linearly along the axis of symmetry of the optimal curve. Substituting for κ we also derive

$$\cos \psi = (y/2c)^2.$$

Note that since the optimal curve bends downwards, its second derivative is negative. This is why, by the usual sign conventions, curvature too is negative. Thus we will use the equation

$$-c\kappa = \sqrt{\cos \psi}$$

between $\psi = +\pi/2$ at the left end and $\psi = -\pi/2$ at the right end of the curve.

Differential Equation for the Curve

The curvature is the rate of turning as one goes along the curve, that is,

$$\kappa = \frac{d\psi}{ds},$$

so $$-c\frac{d\psi}{ds} = \sqrt{\cos \psi},$$

and consequently, $$s = -c\int \frac{d\psi}{\sqrt{\cos \psi}}.$$

The substitution $\cos \psi = \xi$ leads to a denominator of the form

$$\sqrt{\xi(1-\xi)(1+\xi)},$$

while the substitution $\cos \psi = \xi^2$ leads to a denominator of the form

$$\sqrt{1 - \xi^4}.$$

In each case we are dealing with the integral of a rational function of ξ and the square root of a cubic or quartic polynomial in ξ. This means that the answer can be expressed as an elliptic integral [pg. 16, 9; pg. 833, 10; pg. 904, 11; pg. 589, 12].

We can actually just look up the result directly [using pg. 154, 11; also see Appendix] and find the solution

$$s = \sqrt{2}\, c\, F(\cos^{-1}\sqrt{\cos\psi},\ 1/\sqrt{2}),$$

where F is the incomplete elliptic integral of the first kind.[1] The constant of integration has been chosen so that $\psi=0$ corresponds to $s=0$. The result came out positive because the integration goes from $\psi=0$ to negative values of ψ. For the half of the optimal curve in the negative quadrant, a minus sign must be attached. Some readers may notice the similarity of the solution found above, to the equation for a pendulum [pg. 28, 9], swinging from $-\pi/2$ to $+\pi/2$ (where ψ is the angle from the vertical, while s corresponds to the time).

We now have the equation of the curve sought after in Whewell form [pg. 4, 14], namely arc length as a function of tangential angle. We can immediately also rewrite it in Cesáro form [pg. 4, 14], namely as a relation between arc length and curvature:

$$s = \sqrt{2}\, c\, F(\cos^{-1}(-\, c\, \kappa),\ 1/\sqrt{2}).$$

Both of the forms given above are intrinsic equations for the curve [pg. 40, 2].

We can easily compute the length of the curve from an initial point at the top, where $\psi=0$, to the point on the x-axis, where $\psi = -\pi/2$, since

$$F(\pi/2,\ 1/\sqrt{2}) = K(1/\sqrt{2})$$

where K is the complete elliptic integral of the first kind. Now [pg. 909, 11],

$$K(1/\sqrt{2}) = K(\sin(\pi/4)) = \sqrt{2}\int_0^1 \frac{dt}{\sqrt{1-t^4}} = \frac{\Gamma(1/4)^2}{(4\sqrt{\pi})},$$

where Γ is the gamma function [pg. 821, 10; pg. 933, 11; pg. 255, 12]. There is an infinite product [pg. 938, 11] for $\Gamma(1/4)^4$

$$\Gamma(1/4)^4 = 16\,\pi^2 \prod_{k=1}^{\infty} \frac{(4k-1)^2[(4k+1)^2-1]}{(4k+1)^2[(4k-1)^2-1]}$$

which gives us the numerical value

$$\Gamma(1/4) = 3.6256099082...$$

The arc length is finally,

$$\mathcal{S} = \frac{c}{2}\frac{\Gamma(1/4)^2}{\sqrt{2\pi}}.$$

---

1. Note that some authors [pg. 833, 10] list the arguments of the incomplete elliptic functions in the reverse order of that shown here.

Cartesian Form of the Solution

For many purposes it is more convenient to express the solution as a relationship between the x and y coordinates.

We note that
$$\frac{dx}{ds} = \cos\psi \text{ and } \frac{dy}{ds} = \sin\psi,$$

and remember that
$$-c\frac{d\psi}{ds} = \sqrt{\cos\psi}.$$

By the chain-rule for differentiation
$$\frac{dx}{d\psi} = \frac{dx}{ds} / \frac{d\psi}{ds} = -c\sqrt{\cos\psi},$$

and
$$\frac{dy}{d\psi} = \frac{dy}{ds} / \frac{d\psi}{ds} = -c\sin\psi / \sqrt{\cos\psi}.$$

The latter equation is easy to integrate using the substitution $z = \cos\psi$.

$$y = c\int\frac{dz}{\sqrt{z}} = 2c\sqrt{z},$$

so that
$$(y/2c) = \sqrt{\cos\psi},$$

where the constant of integration was chosen so that $y=0$ when $\psi=-\pi/2$. The reader may also recall that in the previous section this result was found directly as the limit of height of the multi-arc approximation.

The integral for x is a little harder,
$$x = c\int\sqrt{\cos\psi}\, d\psi.$$

This can be expressed as the difference of two incomplete elliptic integrals [using pg. 156, 11; also see Appendix],[1]

$$x = \sqrt{2}\,c[\,2\,E(\cos^{-1}\sqrt{\cos\psi}, 1/\sqrt{2}) - F(\cos^{-1}\sqrt{\cos\psi}, 1/\sqrt{2})\,],$$

where the constant of integration was chosen so that $x=0$ when $\psi=0$.[2] Finally then,

$$x = \sqrt{2}\,c[\,2\,E(\cos^{-1}(y/2c), 1/\sqrt{2}) - F(\cos^{-1}(y/2c), 1/\sqrt{2})\,].$$

An alternate way to obtain the same result is to note that x and y are related by the differential equation

$$\frac{dy}{dx} = -\frac{\sqrt{1-(y/2c)^4}}{(y/2c)^2},$$

---

1. Note that E denotes two things in this paper. When the letter appears with two argument it signifies the *incomplete* elliptic integral of the second kind.
2. Note again that some authors [pg. 833, 10] list the arguments of the incomplete elliptic functions in the reverse order of that shown here.

since $$\frac{dy}{dx} = \tan\psi = \frac{\sqrt{1-\cos^2\psi}}{\cos\psi}.$$

In any case, for $y = 0$,

$$x = \sqrt{2}\, c\, [\, 2\, E(1/\sqrt{2}) - K(1/\sqrt{2})\,],$$

where K and E are the complete elliptic integrals of the first and second kind respectively.

Using Legendre's identity [pg. 25, 9; pg. 836, 10; pg. 907, 11; pg. 591, 12]

$$E(k)K(k') + E(k')\, K(k) - K(k)K(k') = \pi/2,$$

where $$k^2 + k'^2 = 1,$$

we get [pg. 25, 9] $$2\, E(1/\sqrt{2}) = [\, \pi/2 + K^2(1/\sqrt{2})\,] \,/\, K(1/\sqrt{2}),$$

so that $$[\, 2\, E(1/\sqrt{2}) - K(1/\sqrt{2})\,] = (\pi/2) \,/\, K(1/\sqrt{2}).$$

So the width of the curve is $$\mathcal{W} = c\,\frac{(2\pi)^{3/2}}{\Gamma(1/4)^2},$$

and, if we want $\mathcal{W} = 1$, we must have

$$c = \frac{\Gamma(1/4)^2}{(2\pi)^{3/2}} = 0.8346268416...$$

The height of the curve then comes to

$$\mathcal{H} = 2c = 1.669253683...$$

(compare to the two-arc approximation). The minimum radius of curvature, the inverse of the maximum curvature, is

$$r_{\mathrm{min}} = \frac{1}{(\mathcal{H}/2c^2)} = c = 0.8346268416...$$

Thus a circle tangent to the curve at the top is also tangent to the x-axis.

The arc length comes to $$\mathcal{S} = \frac{1}{2}\,\frac{\Gamma(1/4)^4}{(2\pi)^2},$$

or $$\mathcal{S} = 2.188439615... \approx 1.393203929*(\pi/2).$$

Finally, from $$\frac{dE}{dx} = 1/c^2,$$

we get $$\mathcal{E} = 1/c^2,$$

and so $$\mathcal{E} = \frac{(2\pi)^3}{\Gamma(1/4)^4} = 1.435540022... \approx 0.9138931623*(\pi/2).$$

Note that $\mathcal{E}\,\mathcal{S} = \pi$. The curve of least energy is shown in Figure 8.

Extension of the Curve

So far we have considered a finite segment of the optimum curve, extending from a point of zero curvature (where $\psi = -\pi/2$) through a point of maximum curvature (where $\psi = 0$) to a second point of zero curvature (where $\psi = +\pi/2$). Can the curve be extended beyond these points?

It is clear that ψ must remain in the range $[-\pi/2, +\pi/2]$ so that the square root of its cosine remains real. To continue the curve then, the sign of the curvature must change; we must choose the other sign for the square-root. The new segment we obtain has the same shape, of course, as the segment we have found already, just inverted.

The segment we have used so far is just a piece of the infinite periodic wave shown in Figure 9. In Figure 10 we see several curves which correspond to stationary values of the integral and pass through the specified points with the desired orientation. The one on the left is the one which corresponds to a global minimum of the energy. The curves containing n half-cycles have an energy n^2 as large as the one containing a single half-cycle.

The curve of least energy passing through two given points with specified orientation is just a portion of the general curve, suitably translated, rotated, and scaled. This is illustrated in Figure 11. The rotated, translated and scaled curves form a four parameter family.

Variational Approach

We are trying to find the curve for which $\quad \mathcal{E} = \int \kappa^2 \, ds$

is minimal. This integral can also be written in the form

$$\mathcal{E} = \int \kappa^2 [1 + (y')^2]^{1/2} \, dx,$$

or, since

$$\kappa = \frac{y''}{[1 + (y')^2]^{3/2}},$$

as

$$\mathcal{E} = \int \frac{(y'')^2}{[1 + (y')^2]^{5/2}} \, dx .$$

This is of the form

$$\int \mathcal{F}(x, y, y', y'') \, dx,$$

and the calculus of variation [pg. 119, 15; pg. 190, 16; pg. 198, 17] teaches us that for a stationary value of the integral,

$$\mathcal{F}_y - \frac{d}{dx}\mathcal{F}_{y'} + \frac{d^2}{dx^2}\mathcal{F}_{y''} = 0,$$

where $\mathcal{F}_y$, $\mathcal{F}_{y'}$, and $\mathcal{F}_{y''}$ are the partial derivatives of $\mathcal{F}$ with respect to y, y', and y'' respectively:

$$\mathcal{F}_y = 0,$$

$$\mathcal{F}_{y'} = -\frac{5\,y'(y'')^2}{[1 + (y')^2]^{7/2}},$$

$$\mathcal{F}_{y''} = \frac{2\,y''}{[1 + (y')^2]^{5/2}}.$$

Since $\mathcal{F}_y = 0$,
$$-\frac{d}{dx}\Big[\mathcal{F}_{y'} - \frac{d}{dx}\mathcal{F}_{y''}\Big] = 0,$$

and integrating, we get
$$-\mathcal{F}_{y'} + \frac{d}{dx}\mathcal{F}_{y''} = A,$$

where A is an arbitrary constant. In the above we have closely followed the approach taken by Mehlum [pgs. 157&189, 2; pg. 43, 6].

Also
$$\frac{d}{dx}\mathcal{F}_{y''} = \frac{2\,y'''}{[1 + (y')^2]^{5/2}} - \frac{10\,y'(y'')^2}{[1 + (y')^2]^{7/2}},$$

so we get
$$\frac{2y'''}{[1 + (y')^2]^{5/2}} - \frac{5y'(y'')^2}{[1 + (y')^2]^{7/2}} = A.$$

Now
$$\frac{dy''}{dy'} = \frac{dy''}{dx} \Big/ \frac{dy'}{dx} = y''' / y'',$$

so
$$y''' = y''\frac{dy''}{dy'}.$$

In addition
$$\frac{d}{dy'}\frac{1}{2}(y'')^2 = y''\frac{dy''}{dy'},$$

and
$$\frac{d}{dy'}\frac{1}{[1 + (y')^2]^{5/2}} = -\frac{5y'}{[1 + (y')^2]^{7/2}}.$$

Using these results in the equation above,

$$\frac{d}{dy'}\frac{(y'')^2}{[1 + (y')^2]^{5/2}} = A,$$

and, integrating, we get
$$\frac{(y'')^2}{[1 + (y')^2]^{5/2}} = Ay' + B,$$

where B is a second arbitrary constant.

Returning for a moment to the integral of the square of curvature, we see that

$$\int \kappa^2 ds = \int [Ay' + B]\, dx = \int A\, dy + \int B\, dx$$
$$= A(y_1 - y_0) + B(x_1 - x_0)$$

for a curve which starts at (x_0, y_0) and ends at (x_1, y_1).

Now
$$\frac{(y'')^2}{[1 + (y')^2]^{5/2}} = \kappa^2 \sqrt{1 + (y')^2},$$

so
$$\kappa^2 \sqrt{1 + (y')^2} = A y' + B.$$

Also,
$$\frac{dy}{dx} = \frac{dy}{ds} \Big/ \frac{dx}{ds} = \dot{y}/\dot{x},$$

so
$$\kappa^2 \sqrt{1 + (\dot{y}/\dot{x})^2} = A(\dot{y}/\dot{x}) + B,$$

or
$$\kappa^2 = A\dot{y} + B\dot{x},$$

since
$$\dot{x}^2 + \dot{y}^2 = 1.$$

Now if
$$\tan \psi = \frac{dy}{dx},$$

then
$$\dot{x} = \cos \psi \text{ and } \dot{y} = \sin \psi .$$

Remembering that
$$\kappa = \frac{d\psi}{ds},$$

we finally see that
$$\frac{d\psi}{ds} = \pm \sqrt{A \sin \psi + B \cos \psi} .$$

Letting
$$A = 1/c^2 \cos \varphi \text{ and } B = -1/c^2 \sin \varphi$$

we get
$$c \frac{d\psi}{ds} = \pm \sqrt{\sin (\psi - \varphi)} .$$

The scale of the curve is dependent on the parameter c, while the rotation in the xy-plane is dependent on the parameter φ.

Altogether we have a four parameter family of curves, since we can also choose an initial point and a direction for the curve. Conversely, we can find a single curve out of this family which passes through any two points with orientation specified at both points.

By the way, if we let the line from the initial point (x_0, y_0) to the final point (x_1, y_1) have length r and direction θ, then

$$\int \kappa^2 ds = (r/c^2) \sin (\theta - \varphi).$$

Summary

We have found the simple equation

$$\pm\, c\,\kappa = \sqrt{\cos(\psi - \varphi)},$$

and solved it to find the equation of the curve of least energy in Whewell form

$$s = \pm\sqrt{2}\, c\, F(\cos^{-1}\sqrt{\cos(\psi-\varphi)}, 1/\sqrt{2}).$$

We also developed a differential equation for the curve in Cartesian coordinates aligned with the axis of symmetry,

$$\frac{dy}{dx} = -\frac{\sqrt{1-(y/2c)^4}}{(y/2c)^2}.$$

The solution of this equation, for given initial conditions, led to

$$x = \sqrt{2}c\,[\,2\,E(\cos^{-1}(y/2c),\ 1/\sqrt{2}) - F(\cos^{-1}(y/2c), 1/\sqrt{2})\,].$$

We considered the curve of least energy connecting the point $(-1,0)$ to the point $(+1,0)$ with vertical initial and final orientations. This curve has minimum radius of curvature

$$c = \frac{\Gamma(1/4)^2}{(2\pi)^{3/2}},$$

rises to a height

$$\mathcal{H} = 2c,$$

has arc length

$$\mathcal{S} = \frac{1}{2}\,\frac{\Gamma(1/4)^4}{(2\pi)^2},$$

and energy

$$\mathcal{E} = \frac{(2\pi)^3}{\Gamma(1/4)^4},$$

or about 91.39% of that of the simple semicircle approximation.

We have also given a method for finding approximations, consisting of circular arcs, to the curve of least energy.

Note that the curve found here is *extensible* [7] in the sense that if the least energy curve with orientation α at A and orientation β at B passes through the point C with orientation γ, then the segments from A to C and from C to B are themselves least energy curves. (This is not true of the two-arc approximation [8]). As a result, such a curve can be computed by a simple, locally connected network.

We have not shown how to find the particular member of the four parameter family of curves which passes through a specified pair of points with specified orientation. Presumably determining the axis of symmetry of the curve would be a helpful first step in this direction. In practical applications the multi-arc approximation method may be suitable in dealing with this problem. We have not discussed how one might compute the curve of least energy passing through three or more points. Here there is no constraint on the direction, but the curvature must be continuous. Nor have we touched upon the extension to curves and surfaces in three dimensions, a topic which Mehlum addresses [pg. 62, 6].

Acknowledgments

Mike Brady and Eric Grimson drew my attention to this problem and provided a number of valuable insights. The assistance of Judi Jones in creating the figures and in dealing with the text justifier is also appreciated.

Appendix

In this appendix we determine two integrals needed in the body of the paper.

To evaluate the integral
$$I_1 = \int_0^{\psi} \frac{d\psi}{\sqrt{\cos\psi}},$$
we substitute
$$\xi^2 = \cos\psi,$$
and obtain
$$I_1 = -2\int_1^{\sqrt{\cos\psi}} \frac{d\xi}{\sqrt{1-\xi^4}}.$$
Next, substituting
$$t^2 = 1 - \xi^2,$$
we get
$$I_1 = \sqrt{2}\int_0^{\sqrt{1-\cos\psi}} \frac{dt}{\sqrt{1-t^2}\sqrt{1-t^2/2}},$$
and since
$$\sin^{-1}\sqrt{1-\cos\psi} = \cos^{-1}\sqrt{\cos\psi},$$
we finally get
$$\int_0^{\psi} \frac{d\psi}{\sqrt{\cos\psi}} = \sqrt{2}\,F(\cos^{-1}\sqrt{\cos\psi},\, 1/\sqrt{2}).$$

To evaluate the integral
$$I_2 = \int_0^{\psi} \sqrt{\cos\psi}\, d\psi,$$
we again substitute
$$\xi^2 = \cos\psi,$$
and obtain
$$I_2 = -2\int_1^{\sqrt{\cos\psi}} \frac{\xi^2}{\sqrt{1-\xi^4}}\, d\xi.$$
Now
$$\frac{\xi^2}{\sqrt{1-\xi^4}} = \frac{1+\xi^2-1}{\sqrt{1+\xi^2}\sqrt{1-\xi^2}} = \frac{\sqrt{1+\xi^2}}{\sqrt{1-\xi^2}} - \frac{1}{\sqrt{1-\xi^4}}.$$
The integral thus can be split into two parts, the second of which we have already evaluated.

To evaluate $$I_2 - I_1 = -2\int_1^{\sqrt{\cos\psi}} \frac{\sqrt{1+\xi^2}}{\sqrt{1-\xi^2}}\,d\xi\,,$$

we substitute as before $$t^2 = 1 - \xi^2$$

and obtain $$I_2 - I_1 = 2\sqrt{2}\int_0^{\sqrt{1-\cos\psi}} \frac{\sqrt{1-t^2/2}}{\sqrt{1-t^2}}\,dt\,,$$

and so $$I_2 - I_1 = 2\sqrt{2}\,E(\cos^{-1}\sqrt{\cos\psi},\, 1/\sqrt{2})\,,$$

since $$\sin^{-1}\sqrt{1-\cos\psi} = \cos^{-1}\sqrt{\cos\psi}\,.$$

Finally then,

$$\int_0^{\psi} \sqrt{\cos\psi}\,d\psi = \sqrt{2}\,[\,2\,E(\cos^{-1}\sqrt{\cos\psi},\, 1/\sqrt{2}) - F(\cos^{-1}\sqrt{\cos\psi},\, 1/\sqrt{2})\,].$$

The corresponding definite integrals from $\psi = 0$ to $\psi = \pi/2$ can be expressed in terms of complete elliptic integrals of the first and second kind [pg. 91, 9].

References

1. Den Hartog, J. P., *Strength of Materials*, Dover Publications, New York, 1949.

2. Faux, Ivor D. & Pratt, Michael J., *Computational Geometry for Design and Manufacture*, Ellis Horwood Limited, Chichester, England, 1979.

3. de Boor, Carl, *A Practical Guide to Splines*, Springer Verlag, New York, 1978.

4. Giloi, Wolfgang K., *Interactive Computer Graphics: Data Structures, Algorithms, Languages*, Prentice-Hall, Englewood Cliffs, New Jersey, 1978.

5. Newman, William M. & Sproull, Robert F., *Principles of Interactive Computer Graphics*, McGraw-Hill, New York, 1979.

6. Mehlum, Even, *Curve and Surface Fitting Based on Variational Criteriae for Smoothness*, Central Institute for Industrial Research, Oslo, Norway, 1969.

7. Ullman, Shimon, "Filling-in the Gaps: The Shape of Subjective Contours and a Model for their Generation," *Biological Cybernetics*, Vol. 25, pp. 1–6, 1976.

8. Brady, M., Grimson, W. E. L. & Langridge, D. J.. "Shape Encoding and Subjective Contours," *Proc. First Annual National Conf. Artificial Intelligence*, Stanford University, 1980 August 18–21, pp. 15–17.

9. Bowman, Frank, *Introduction to Elliptic Functions with Applications*, Dover Publications, New York, 1961.

10. Korn, Granino A. & Korn, Theresa M., *Mathematical Handbook for Scientists and Engineers*, McGraw-Hill, New York, 1968.

11. Gradshteyn, I. S. & Ryzhik, I. M., *Tables of Integrals, Series, and Products*, Academic Press, New York, 1980.

12. Abramowitz, Milton & Stegun, Irene A., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Washington, 1972.

13. Thomas, George B. & Finney, Ross L., *Calculus and Analytic Geometry*, Addison-Wesley, Reading, Massachusetts, 1980.

14. Lawrence, J. Dennis *A Catalog of Special Plane Curves*, Dover Publications, New York, 1972.

15. Forsyth, A. R., *Calculus of Variation*, Dover Publications, New York, 1960.

16. Courant, R. & Hilbert, D., *Methods of Mathematical Physics*, Volume I, Interscience Publishers, New York, 1953.

17. Hildebrand, Francis B., *Methods of Applied Mathematics*, Prentice-Hall, Englewood Cliffs, New Jersey, 1965.

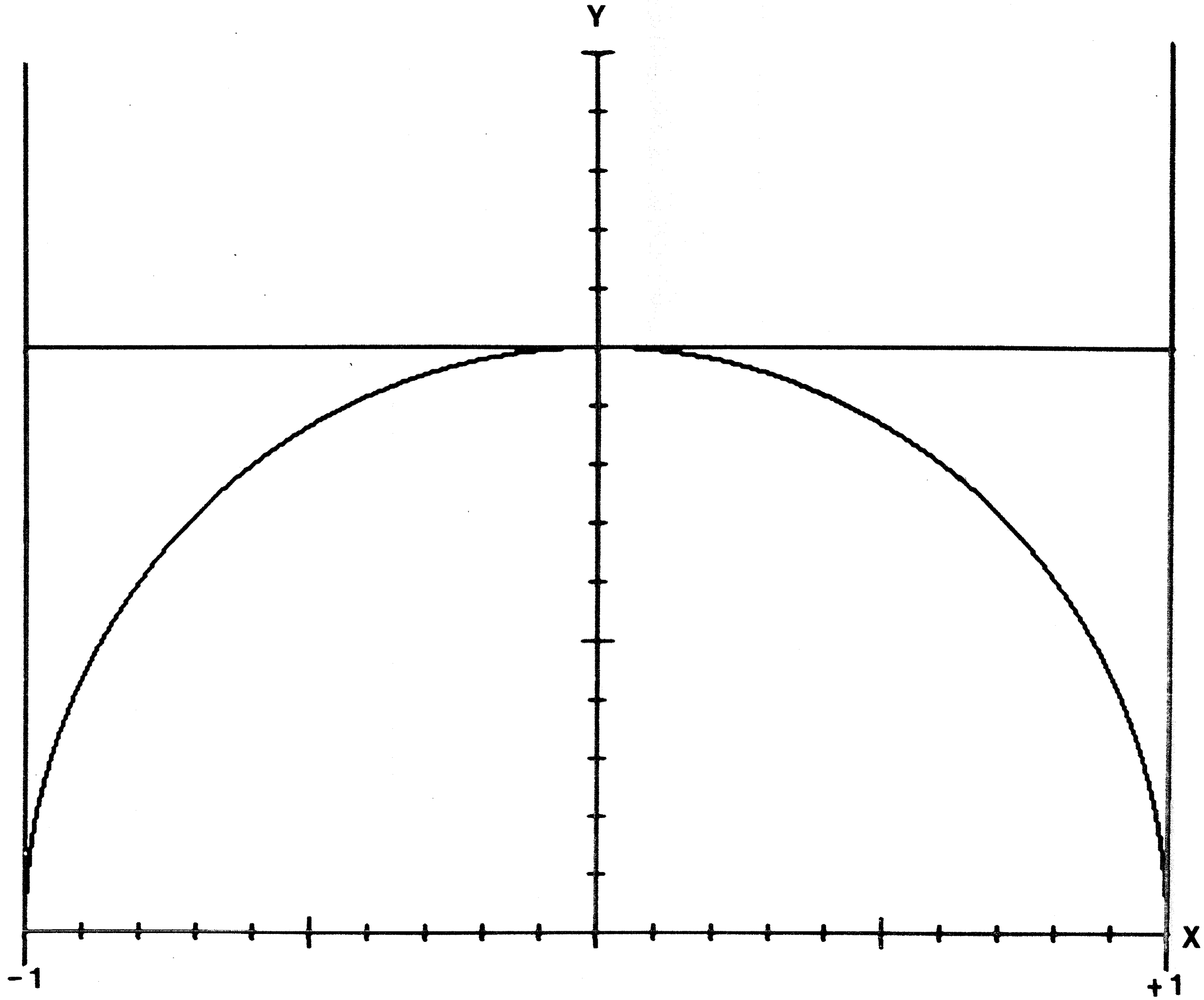

Figure 1: A semicircle connects the points $(-1,0)$ and $(+1,0)$ and has vertical orientation at these points. The arc in the first quadrant has length $\pi/2$ and energy $\pi/2$.

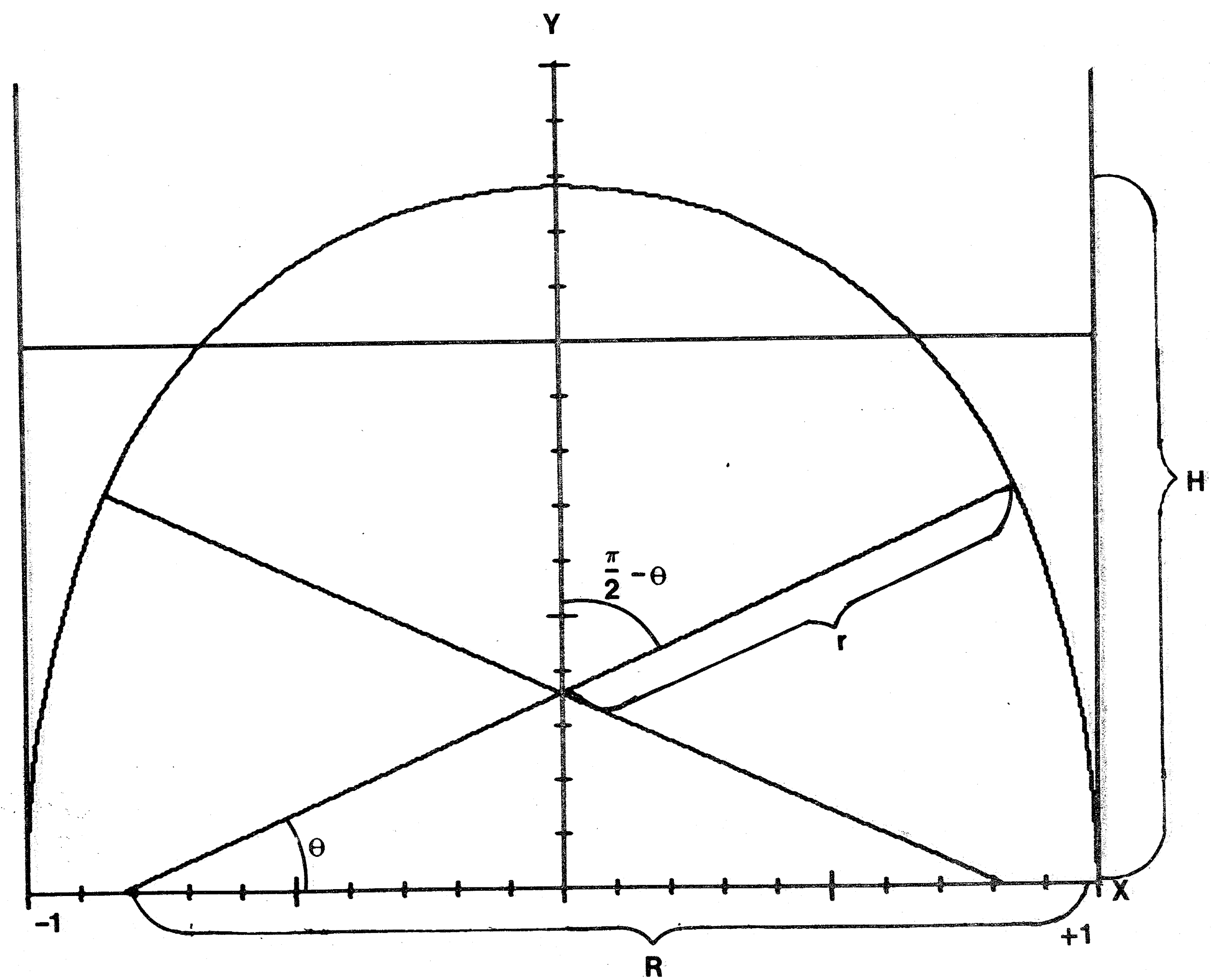

Figure 2: Two circular arcs can be spliced together so that they are tangent at their point of contact. The resulting smooth curve is longer but has less energy than the semicircle.

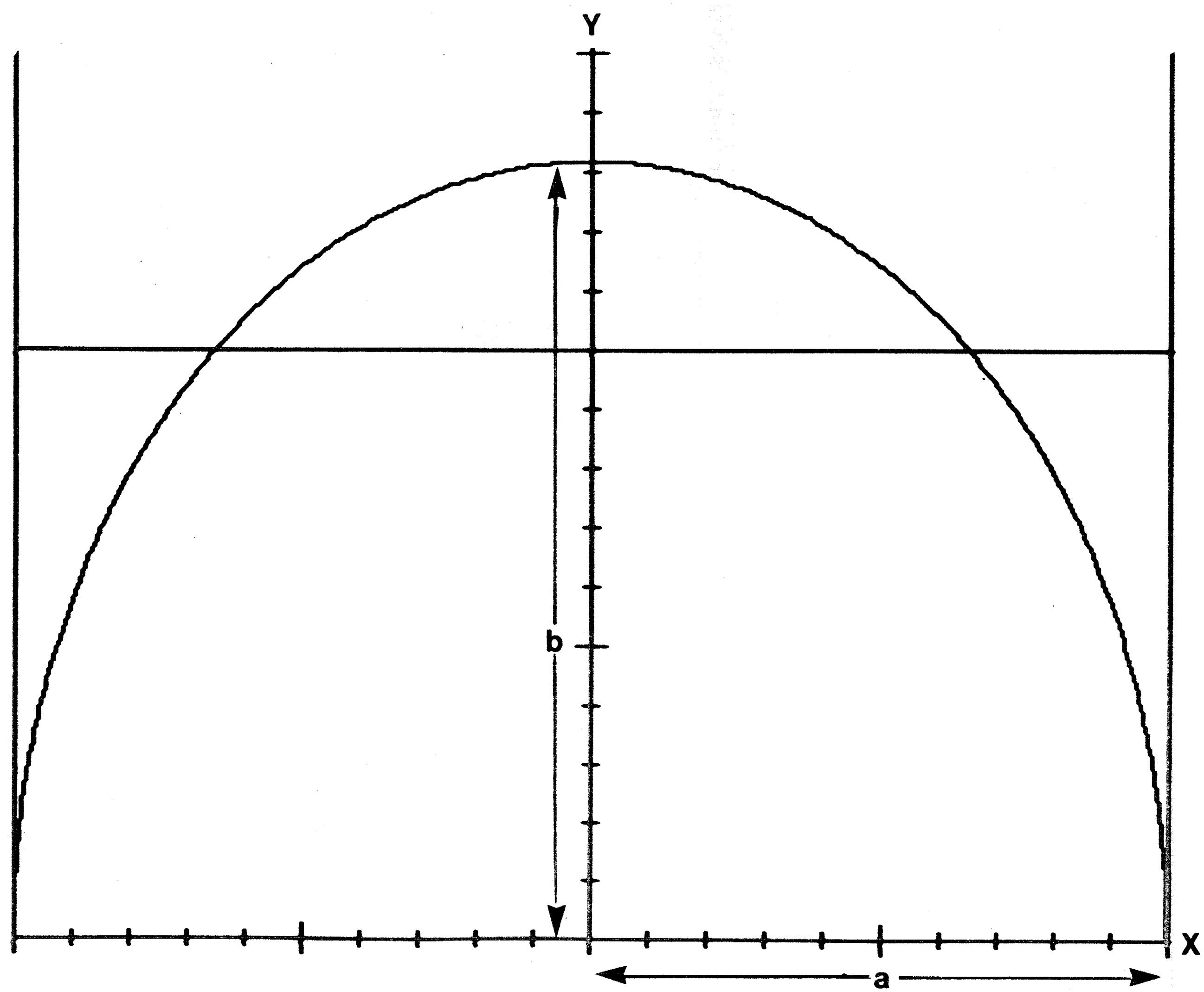

Figure 3: An ellipse with major axis, b, oriented vertically and minor axis, a, oriented horizontally. The eccentricity for which the energy is minimal can be found by differentiation.

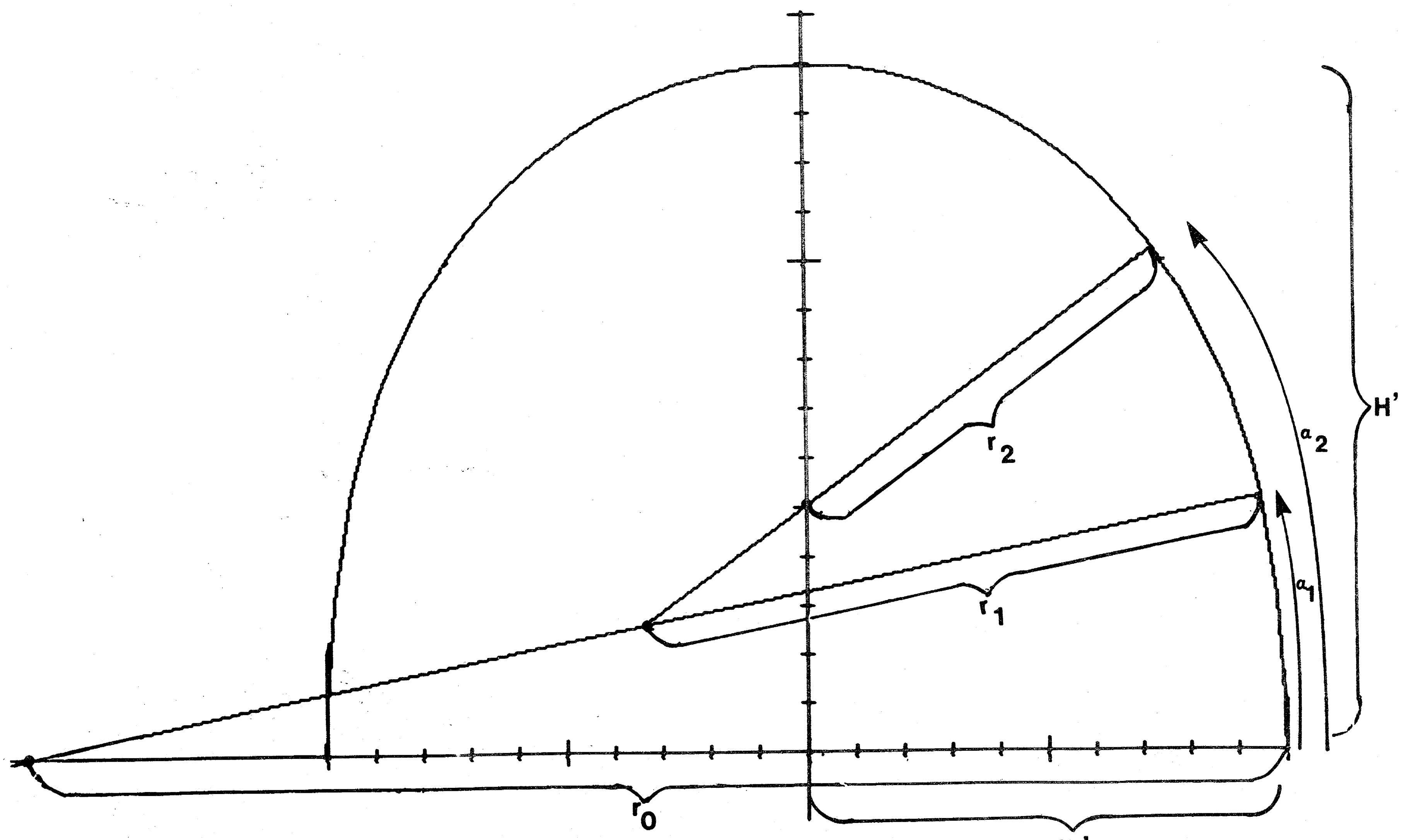

Figure 4: A curve constructed out of *n* circular arcs. Each arc contributes to the total arc length and the total energy.

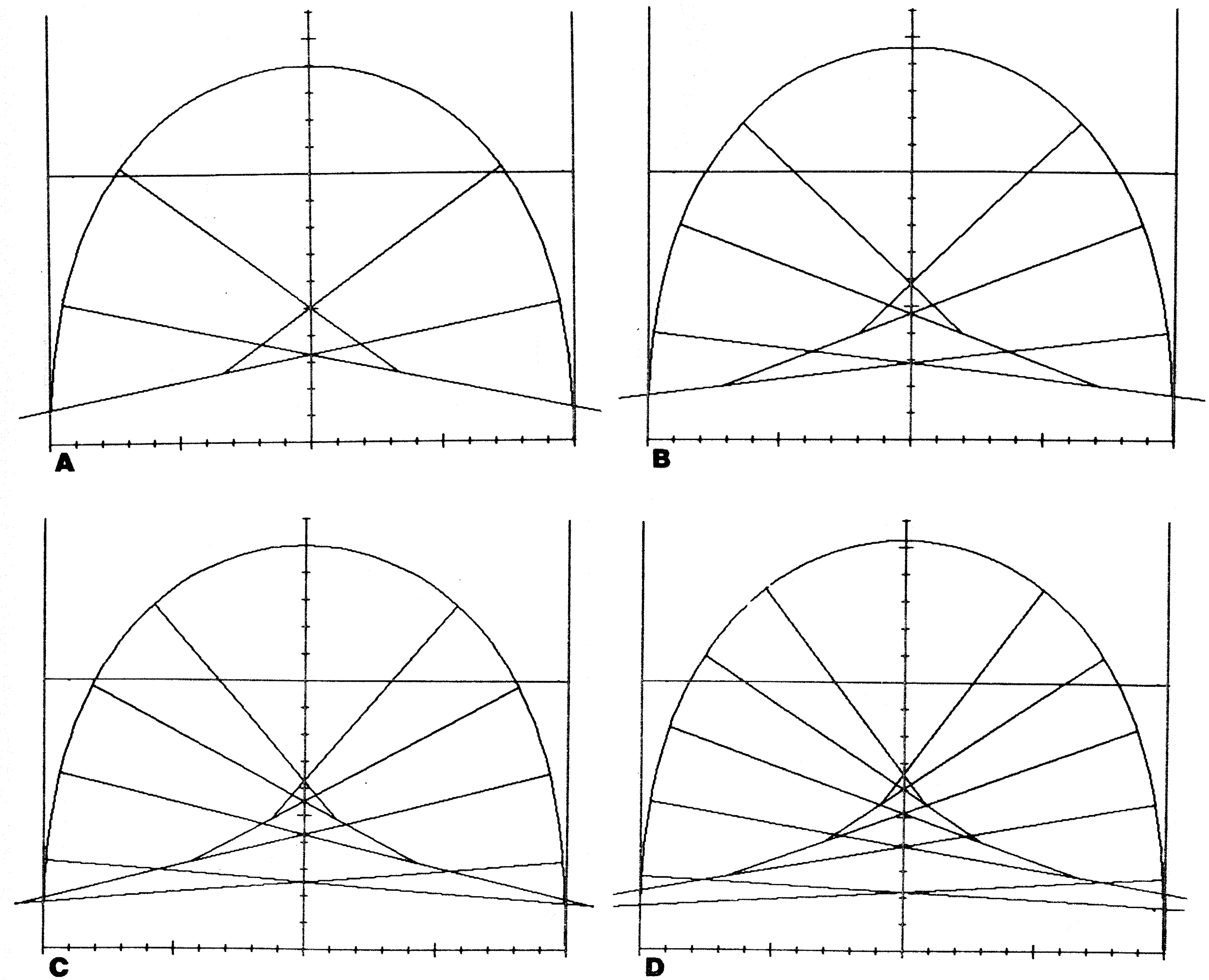

Figure 5: Optimum multi-arc solutions (A) 3 arcs, (B) 4 arcs, (C) 5 arcs, and (D) 6 arcs.

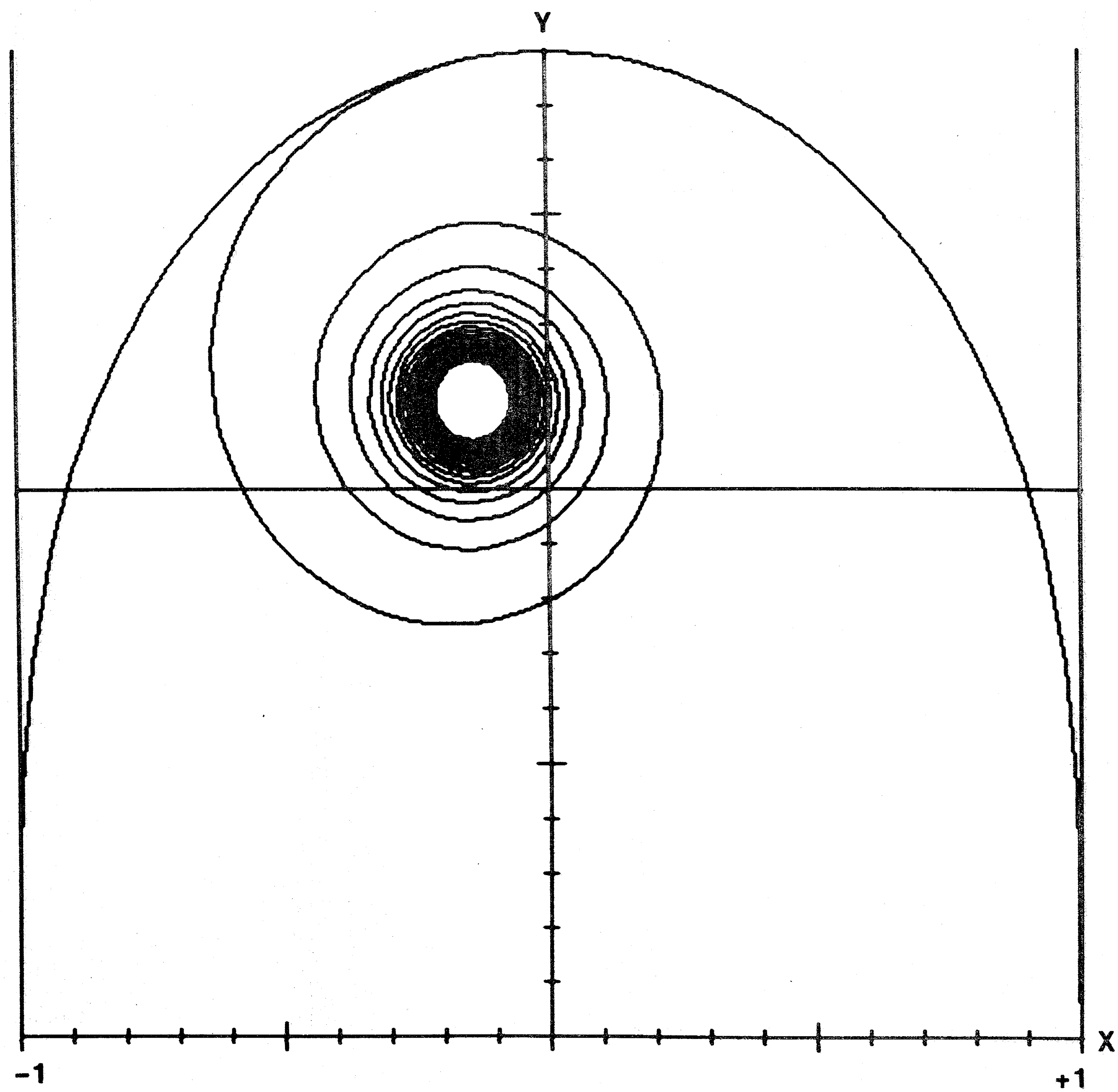

Figure 6: A curve constructed from part of the Cornu Spiral. Here the curvature varies linearly with arc length along the curve.

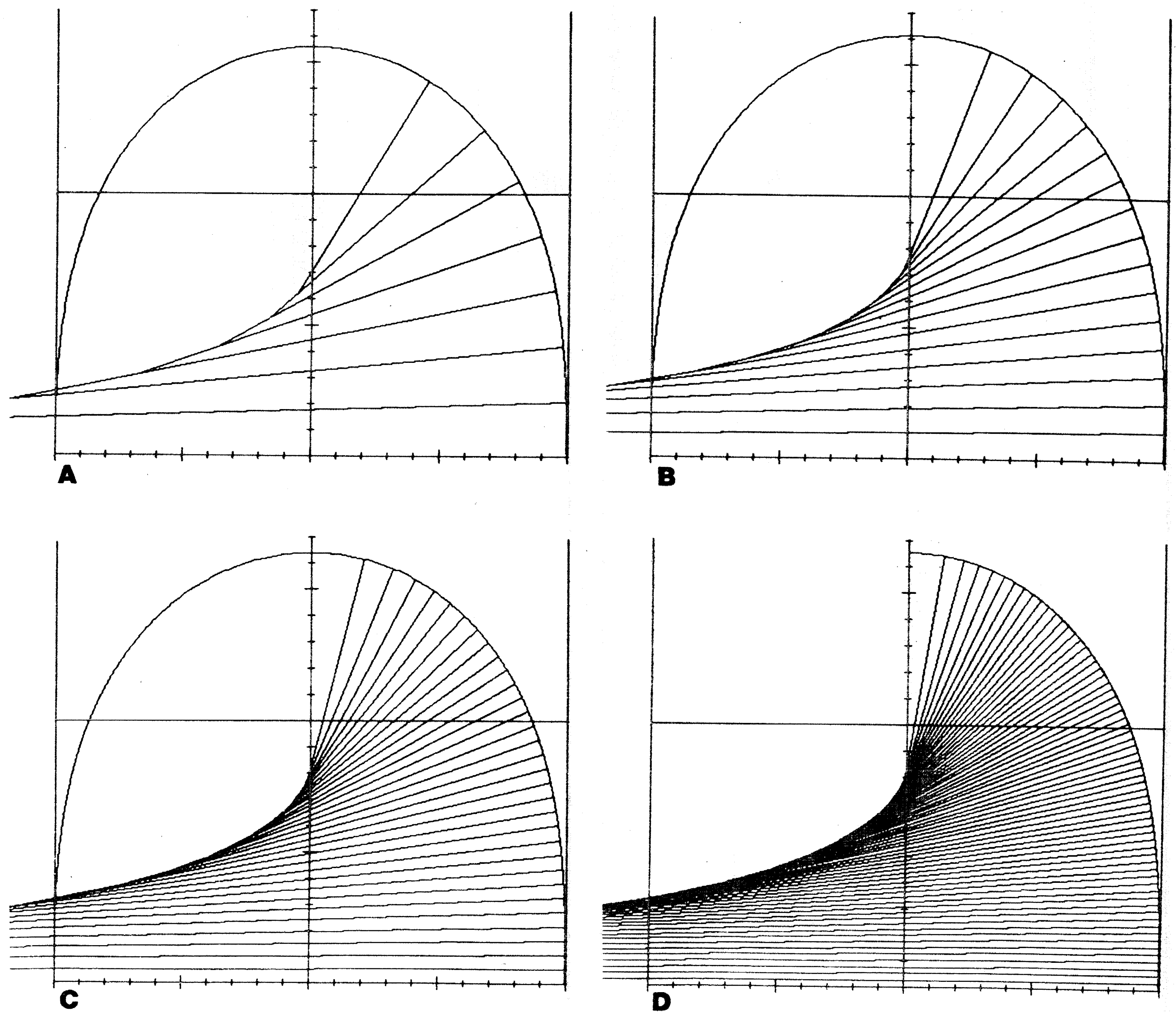

Figure 7: Optimum multi-arc solutions: (A) 8 arcs, (B) 16 arcs, (C) 32 arcs, and (D) 64 arcs.

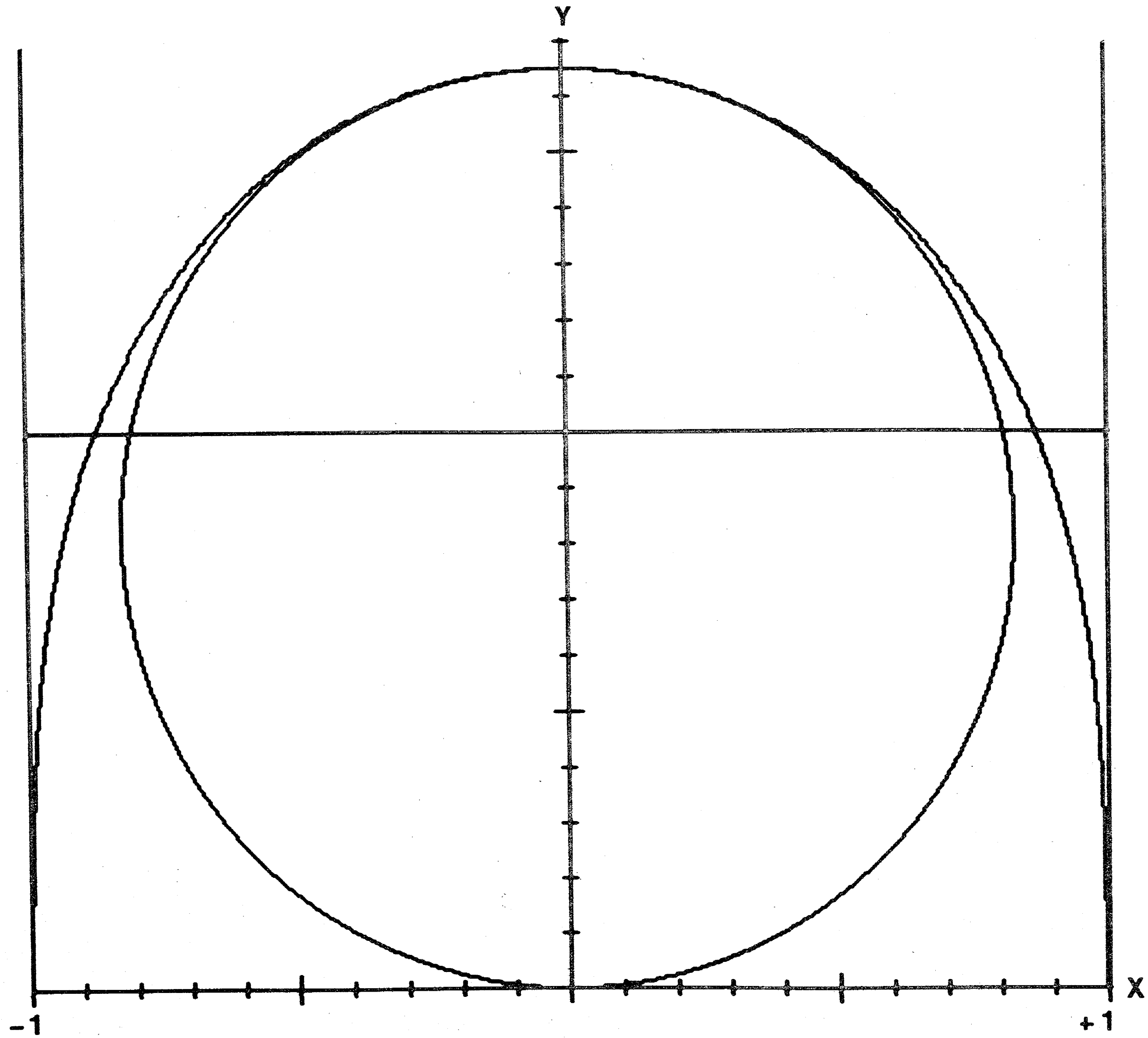

Figure 8: The curve of least energy. Here the curvature varies linearly with distance along the axis of symmetry. A circle tangent at the top is also tangent to the x-axis.

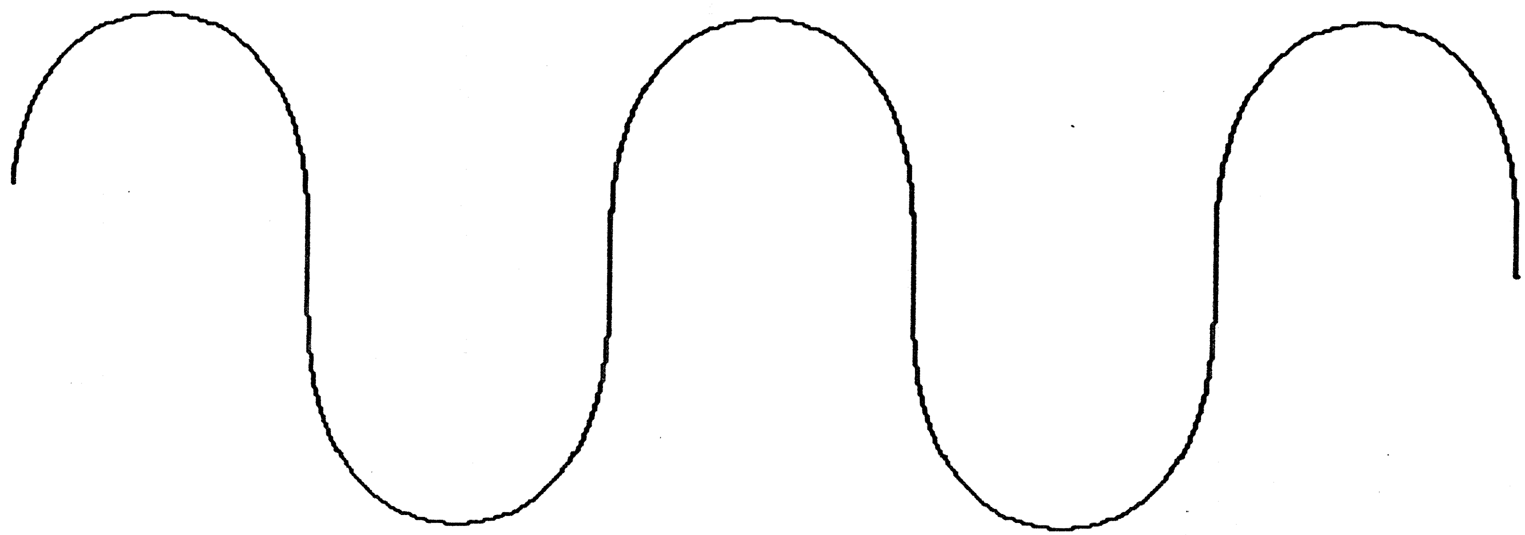

Figure 9: The infinite periodic solution. This curve also describes the motion of a pendulum oscillating between $-\pi/2$ and $+\pi/2$.

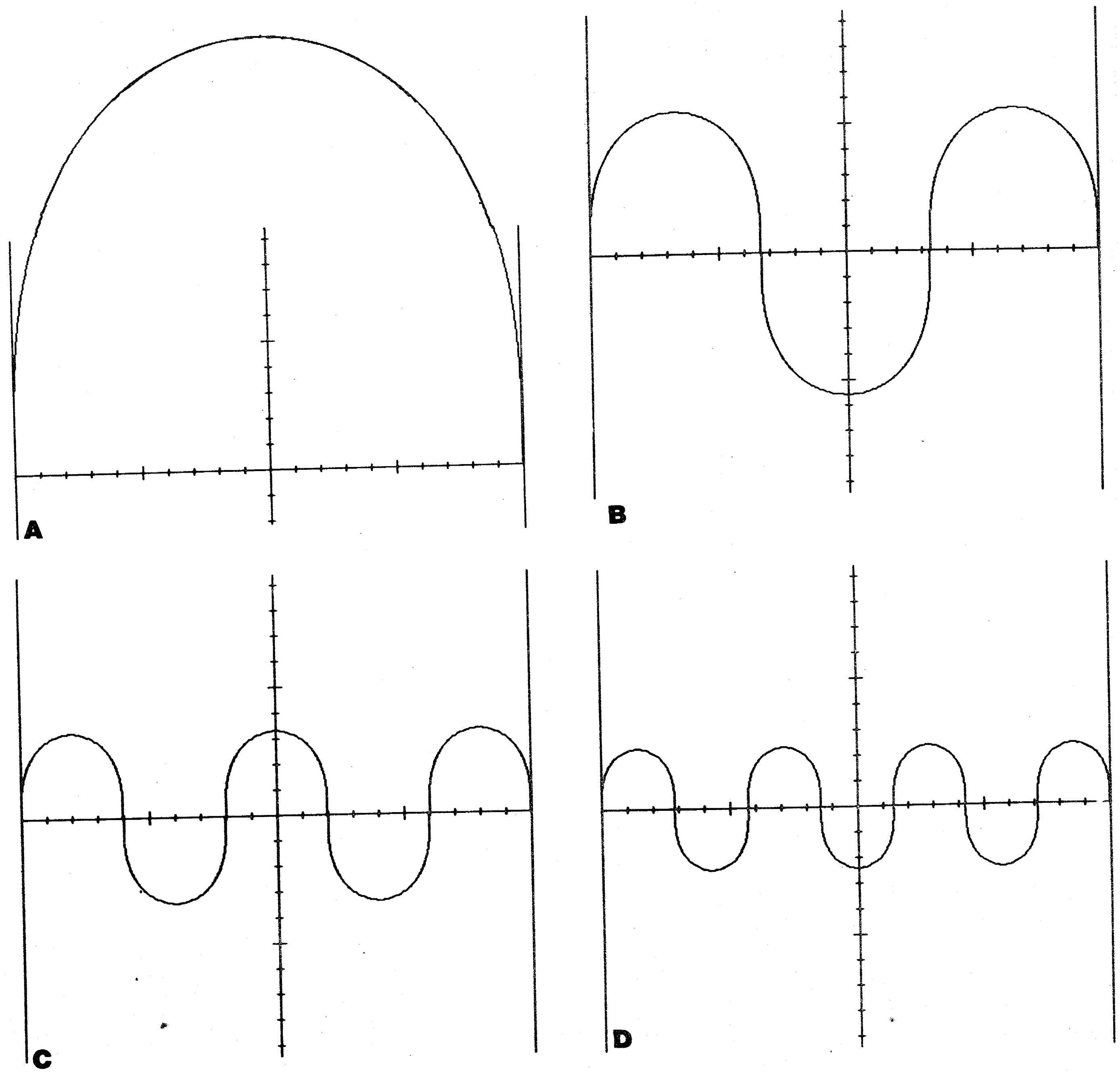

Figure 10: Curves corresponding to stationary values of the integral of the square of the curvature: (A) 1/2 cycle, (B) 3/2 cycles, (C) 5/2 cycles, and (D) 7/2 cycles.

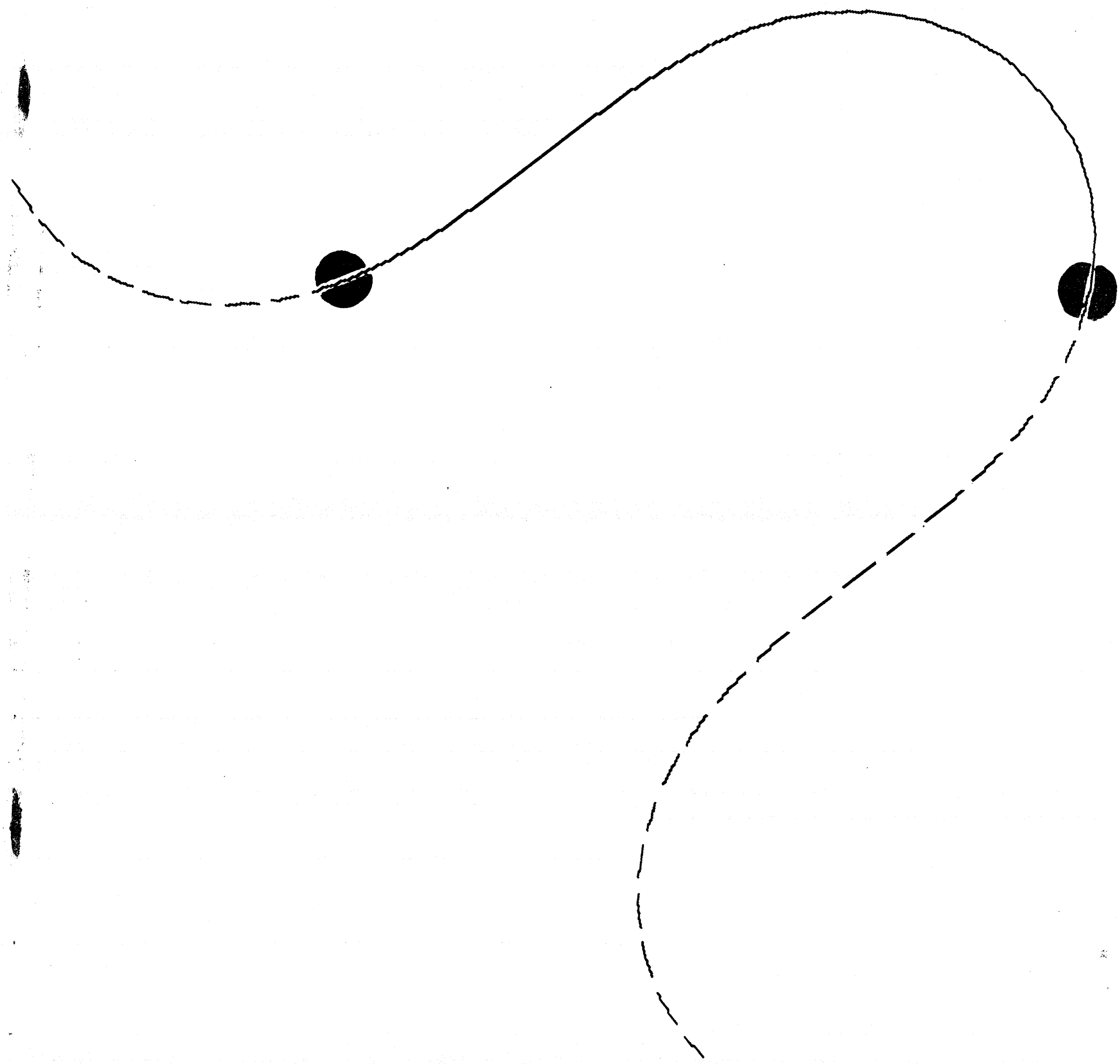

Figure 11: Optimum curve passing through two given points with given orientation. It is a member of a four parameter family of curves obtained by translating, rotating, and scaling the particular curve passing through the points $(-1, 0)$ and $(+1, 0)$ with vertical orientation.